Geometric Regularity of Singularity Models of the Kähler-Ricci Flow

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Outline

Ricci Flow

Definition

A smooth, 1-parameter family of Riemannian metrics $(g_t)_{t\in[0,T)}$ on a closed manifold M^n satisfies Ricci flow if

$$\partial_t g_t = -2Rc(g_t),$$

where $Rc(g_t)$ is the Ricci curvature.

Ricci flow was first used by Richard Hamilton in 1982, to prove that every 3-dim Riemannian manifold with positive Ricci curvature is a space form.

Basic Facts About Ricci Flow

- (PDE Classification) Ricci flow is a second-order nonlinear weakly parabolic system. In harmonic coordinates,

$$\partial_t g_{t,ij} = -2Rc(g_t)_{ij} = \Delta g_{t,ij} + Q_{ij}(g_t, Dg_t).$$

- (Short-time existence/uniqueness) For any smooth Riemannian metric g on M, there is a unique solution $(M^n, (g_t)_{t\in[0,T)})$ of Ricci flow with $g_0 = g$.
- (Reaction-diffusion equation for curvature) If Rm is the curvature tensor, then

$$(\partial_t - \Delta)Rm = Q(Rm),$$

where Q is a quadratic polynomial.

Shrinking Ricci Solitons

Definition

A shrinking gradient Ricci soliton (M^n, g, f) is a Riemannian manifold with $f \in C^\infty(M)$ satisfying

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

- If $\partial_t \varphi_t(x) = \frac{1}{1-t} \nabla f(\varphi_t(x))$, then

$$g(t) = (1-t)\varphi_t^* g$$

  solves the Ricci flow.
- Examples: Einstein manifolds, shrinking cylinder, FIK soliton.
- Frequently model finite-time singularities of Ricci flow.

Conjugate Heat Kernels

Definition (Conjugate heat kernel)

The conjugate heat equation is

$$\Box^* u := (-\partial_t - \Delta + R)u = 0.$$

The conjugate heat kernel based at (x, t) is the function $(y, s) \mapsto K(x, t; y, s)$ satisfying $\lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x$ and $\Box^*_{y,s} K(x, t; y, s) = 0$. Then $d\nu_{x,t;s} = K(x, t; \cdot, s) dg_s$ is a probability measure for each $s < t$.

Definition

The pointed Nash entropy based at (x, t) at the scale r is

$$\mathcal{N}_{x,t}(r^2) := \int_M f_{x,t}(\cdot, t - r^2) d\nu_{x,t;t-r^2} - \frac{n}{2},$$

where $K(x, t; y, s) = (4\pi(t - s))^{-\frac{n}{2}} e^{-f_{x,t}(y,s)}$

Metric Flows

Definition (Bamler 2020)

A metric flow is a set $\mathcal{X}$ with a time function $\mathfrak{t} : \mathcal{X} \to \mathbb{R}$ whose time slices $\mathcal{X}_t := \mathfrak{t}^{-1}(t)$ are equipped with metrics d_t, and for $x \in \mathcal{X}_t$, $s < t$, there are probability measures $\nu_{x,t;s}$ on $\mathcal{X}_s$ such that for $t_1 < t_2 < t_3$, $x \in \mathcal{X}_{t_3}$, and $A \subseteq \mathcal{X}_{t_1}$, we have

$$\nu_{x;t_1}(A) = \int_{\mathcal{X}_{t_2}} \nu_{y;t_2}(A) d\nu_{x;t_2}(y).$$

Smooth Case: $\mathcal{X} = M \times I$, $\mathfrak{t}$ the projection, $d_t = d_{g_t}$,

$$d\nu_{x,t;s} = K(x, t; \cdot, s) dg_s.$$

Definition (Kleiner-Lott 2014)

A Ricci flow spacetime $(\mathcal{R}, \mathfrak{t}, g, \partial_{\mathfrak{t}})$ is an $(n+1)$-manifold $\mathcal{R}$, a smooth function $\mathfrak{t}$, a vector field $\partial_{\mathfrak{t}}$ on $\mathcal{R}$ with $\partial_{\mathfrak{t}}\mathfrak{t} = 1$ and

$$\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2Rc(g).$$

Structure Theory of Metric Flows

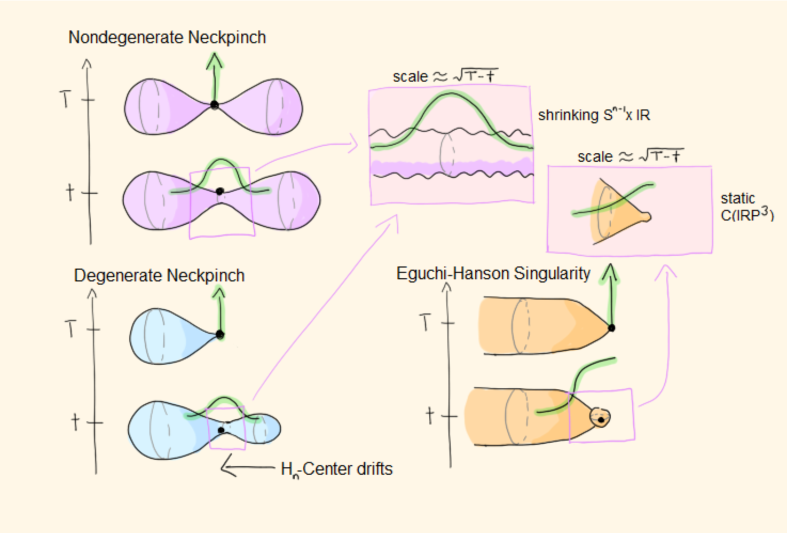
Theorem (Bamler 2020)

Given Ricci flows $(M_j, (g_{j,t})_{t\in[-T_j,0]}, (\nu_{x_j,0;t})_{t\in[-T_j,0]})$ *with* $\mathcal{N}_{x_j,0}(1) \geq -Y$*, a subsequence* $\mathbb{F}$*-converges to some metric flow pair* $(\mathcal{X}, (\nu_{x_\infty;t})_{t\in[-T_\infty,0]})$ *over* $[-T_\infty, 0]$*. Also,*

- $\mathcal{X} = \mathcal{R} \sqcup \mathcal{S}$*, where* $\mathcal{R}$ *has the structure of a smooth Ricci flow spacetime*
- $\mathcal{S}$ *has* P^**-parabolic dimension* $\leq (n-2)$
- *The tangent flow of any point* $x \in \mathcal{X}$ *is a singular shrinking soliton.*

Technicalities: In general, $\mathbb{F}$-convergence does not imply GH convergence, and not every $\mathcal{X}_t$ has singularities of codimension 4.
Tangent flows: If $(M, (g_t)_{t\in[0,T)})$ is a fixed Ricci flow, and $t_j \nearrow T$, set $M_j := M$, $g_{j,t} := (T-t_j)^{-1} g_{T+(T-t_j)t}$, $\nu_{x_j,0;t} := \nu_{p,0;T+(T-t_j)t}$ for $t \in [-(T-T_j)^{-1}T_j, 0)$.

Examples of Singularity Models



Kähler-Ricci Flow

Recall: A Kähler metric on a complex manifold (M, J) is a Riemannian metric g satisfying $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$. Then $\omega := g(J\cdot, \cdot)$ is a symplectic form.

Nice facts about Ricci flow starting from a Kähler metric:

- If $(M, (g_t)_{t\in[0,T)})$ is a Ricci flow and (M, g_0, J) is Kähler, then (M, g_t, J) is Kähler.
- (Cao, Tsuji, Tian-Zhang) The singular time is determined by $[\omega]$.
- (Collins-Tosatti 2013) The singular set is an analytic subset determined by $[\omega]$.
- (Song-Weinkove 2010) The Kähler-Ricci flow can be continued through singularities of Kähler surfaces.
- (Song-Tian 2009) The Kähler-Ricci flow makes sense on projective varieties with "mild" singularities.

The Analytic Minimal Model Program

Issue: Ricci flow should deform a Kähler metric towards a canonical Kähler metric, but a Kähler manifold can only admit a Kähler-Einstein metric if $c_1(X)$ is zero or definite.

Workaround: The minimal model program (MMP) is an algorithm which conjecturally takes any smooth projective variety X to a special variety $\widehat{X}$ via a series of birational transformations. Moreover, $\widehat{X}$ is constructed from varieties with zero or definite c_1.

Conjecture (Song-Tian 2009)

The Kähler-Ricci flow performs the MMP, taking any Kähler metric in a rational cohomology class to a canonical metric (possibly on a different variety) in a continuous way.

Local Model: Soliton Transition

Example (FIK Soliton)

There is a Kähler-Ricci soliton on $\mathcal{O}_{\mathbb{P}^1}(-1)$ which converges to its asymptotic cone at $t = 0$, and then flows into an expanding soliton on $\mathbb{C}^2$ constructed by Cao.

Example (Chi Li's Solitons)

There is Kähler-Ricci soliton on $\mathcal{O}_{\mathbb{P}^m}(-1)^{(n+1)}$ which converges to its asymptotic cone at $t = 0$, and then flows into a soliton on $\mathcal{O}_{\mathbb{P}^n}(-1)^{(m+1)}$ if $n < m$.

Conjecture (J. Song)

A parabolic rescaling at any exceptional point of a birational surgery performed by Ricci flow on projective varieties converges to a shrinking-expanding soliton transition.

The Fano Setting

Suppose $c_1(M) > 0$ and $\omega_0 \in \lambda c_1(M)$, with first singular time $T = 1$. Then $[\omega_t] = (1-t)[\omega_0] = (1-t)\lambda c_1(M)$, so there exists $v \in C^\infty(M)$ satisfying

$$\frac{\omega_t}{1-t} - Rc(\omega_t) = \sqrt{-1}\partial\overline{\partial} v_t.$$

Theorem (Perelman 2006)

For some $C < \infty$, we have

$$|\nabla v_t|^2 + |\Delta v_t| + |R_{g_t}| \leq \frac{C}{1-t},$$

$$diam_{g_t}(M) \leq C\sqrt{1-t}.$$

Theorem (Chen-Wang 2014, Bamler 2015)

For any $x \in M$ and any $t_i \nearrow 1$, $(M_i, |t_i|^{-1} g_{t_i}, x)$ subsequentially converge to a shrinking GRS with singularities of codimension four.

Twisted Ricci Potentials

Suppose $(M,(g_t)_{t\in[0,1)})$ is a compact Kähler-Ricci flow such that $[\omega_1]=\lambda\pi^*\omega_{FS}$ for some $\lambda>0$ and some holomorphic map $\pi:M\to\mathbb{C}P^N$ (holds in projective setting).

Definition (Jian-Song-Tian 2023)

Given a $(1,1)$-form $\theta\in\lambda[\omega_{FS}]$, the corresponding twisted Ricci potential v_t at time t is given (up to constants) by

$$\sqrt{-1}\partial\overline{\partial}v_t=\frac{\omega_t-\pi^*\theta}{1-t}-Rc(\omega_t)$$

Then v solves both elliptic and parabolic equations:

$$\Delta v_t=\frac{n-\mathrm{tr}_{\omega_t}(\theta)}{1-t}-R(\omega_t),$$

$$(\partial_t-\Delta)v=\frac{v-B_0}{1-t}-\frac{1}{1-t}\mathrm{tr}_{\omega_t}(\pi^*\beta).$$

Estimates Near Ricci Vertices

Theorem (Jian-Song-Tian 2023)

There exists $C = C(g_0, \theta) < \infty$ *such that:*
(i) *For all* $t \in [0, 1)$,

$$\frac{|\Delta v|}{v} + \frac{|\nabla v|^2}{v} \leq \frac{C}{1-t}.$$

(ii) *For any Ricci vertex* p_t *associated to* θ,

$$(1-t)|R|(x,t) \leq C\left(1 + \frac{d_t^2(x,p)}{1-t}\right).$$

For any $z_0 \in M$ and any neighborhood U of $\pi(z_0)$ in $\mathbb{C}P^N$, we can choose θ that for $|t-1|$ sufficiently small, each Ricci vertex satisfies $p_t \in \pi^{-1}(U)$.
This is weak control on the location of p_t, and is likely not sharp.

Distortion Estimates

Theorem (H-Jian-Song-Tian 2023)

Let (x_{t_0}, t_0) be a Ricci vertex. Then, for all $x, y \in B(x_{t_0}, t_0, D)$ and $|t - t_0| \leq \alpha(D)(1 - t_0)$, we have

$$d_t(x,y) - C(D)\sqrt{|t-t_0|} \leq d_{t_0}(x,y) \leq d_t(x,y) + C(D)\sqrt{|t-t_0|}$$

Remark (Scale Invariance)

The above estimate is invariant under parabolic zooming in.

Proof in the easy case: Let (u_t) solve the heat flow with $u_{t_0} = d_{t_0}(\cdot, x)$. For $t \geq t_0$, Heat kernel estimates give $u_t(x) \leq C\sqrt{|t-s|}$, $|u_t(y) - d_{t_0}(x,y)| \leq C\sqrt{|t-s|}$, and the maximum principle gives $|\nabla u_t| \leq 1$, so

$$d_t(x,y) \geq u_t(y) - u_t(x) \geq d_{t_0}(x,y) - C\sqrt{|t-s|}.$$

Challenges with Distortion Upper Bound

Proof idea of the hard case: Let (u_t) solve the backwards heat flow with $u_{t_0} = d_{t_0}(\cdot, x)$. For $t < t_0$, heat kernel estimates give

$$|u_t(x)| + |u_t(y) - d_{t_0}(x, y)| \leq C\sqrt{|t - s|},$$

and we know $|\nabla u_{t_0}| \leq 1$.

What goes wrong: $(-\partial_t - \Delta)|\nabla u| \leq C|\nabla u| \cdot |Rc|$, but $|Rc|$ is not controlled. The methods of Chen-Wang and Bamler-Zhang rely on **global** control of $|R|$.

A New Differential Harnack Inequality

Theorem (Zhang-Zhu 2018)

Suppose $(M^n, (g_t)_{t\in[0,T]})$ *is a Ricci flow satisfying* $|R_{g_t}| \leq R_0$, *and* $(u_t)_{t\in[t_0,T]}$ *is a positive solution to the heat equation. Then*

$$-\frac{\Delta u}{u} + \frac{1}{2}\frac{|\nabla u|^2}{u^2} \leq \frac{C}{t-t_0}(R_0+1)^2.$$

This can be localized, and the dependence on R improved, but is not optimal in the Kähler setting.

Theorem (H.-Jian-Song-Tian 2023)

Suppose $(u_t)_{t\in[t_0,1-\epsilon]}$ *is a positive solution to the heat equation, and* v *is the twisted Ricci potential. Then*

$$-\frac{\Delta u}{u} + \frac{1}{2}\frac{|\nabla u|^2}{u^2} \leq C(\epsilon,\theta)\left(\frac{1}{t-t_0}+v\right).$$

Weighted Integral Estimates

Proposition (H.-Jian-Song-Tian 2023)

If $t_0, t_1 \in [0, 1-\epsilon]$, $(u_t)_{t\in[t_0,t_1]}$ *solves* $\partial_t u_t = -\Delta u_t$, *and satisfies* $|\nabla u_{t_1}| \leq 1$, *supp*$(|\nabla u_{t_1}|) \subseteq B(x_{t_1}, t_1, D)$, *then we have*

$$|\nabla u|^2(y, t_0) \leq \exp\left(C\sqrt{t_1 - t_0}(D^2 + 1)\right)$$

for all $y \in M$.

Main idea of proof: Show that

$$\frac{d}{dt}\log\left(\int_M |\nabla u|^2(x,t) e^{A\sqrt{t-t_0}v(x,t)} K(x,t;y,t_0)dg_t(x)\right) \geq -\frac{C}{\sqrt{t-t_0}}.$$

Using the Differential Harnack

Setting
$K(x,t) := K(x,t;y,t_0)$,
$\Phi(t) := \int_M |\nabla u|^2(x,t) e^{A\sqrt{t-t_0}v(x,t)} K(x,t)dg_t(x)$,

$$\begin{aligned}\Phi'(t) \geq &\int_M \left(|\nabla\nabla u|^2 + |\nabla\overline{\nabla} u|^2 - 2\nabla\overline{\nabla} v(\nabla u, \overline{\nabla} u)\right) e^{A\sqrt{t-t_0}v} K dg_t \\ &- C\Phi(t) + \int_M \left(\frac{A}{2\sqrt{t-t_0}} - C\right) |\nabla u|^2 v e^{A\sqrt{t-t_0}v} K dg_t \\ &- 2\mathrm{Re}\int_M \langle \nabla e^{A\sqrt{t-t_0}v}, |\nabla u|^2 \overline{\nabla} K\rangle dg_t.\end{aligned}$$

The last term can be integrated by parts to produce

$$2\int_M |\nabla u|^2 (\partial_t \log K) e^{A\sqrt{t-t_0}v} K dg_t \geq \int_M |\nabla u|^2 |\nabla \log K|^2 e^{A\sqrt{t-t_0}v} K dg_t - \text{sim}.$$

Estimating the Size of the Almost-Singular Set

Proposition (H.-Jian-Song-Tian 2023)

For any $p \in (0,4)$, there exist $E_p = E_p(D) < \infty$ and $\overline{r} = \overline{r}(D) > 0$ such that the following hold for all $x \in B(p_t, t, D)$ and $r \in (0, \overline{r}]$, $s \in (0,1]$:

$$|\{r_{Rm}(\cdot, t) < sr\} \cap B(x,t,r)|_{g_t} \leq E_p s^p r^{2n}.$$

Remark (Bamler's results)

Bamler (2016) proved the above result assuming bounded scalar curvature, and Bamler (2020) showed a spacetime version of the above result with no curvature assumptions.

Proof idea: Contradiction compactness argument – below the Type-I scale, the flows look static; apply Bamler's estimates.

Singularity Models of Projective KRF

Suppose $(M_i, (g_{i,t})_{t\in[-T_i,0]}, (\nu_{p_i,0;t})_{t\in[-T_i,0]})$ is a sequence of Type-I rescalings of a projective Kähler-Ricci flow $\mathbb{F}$-converging to a future-continuous metric flow $(\mathcal{X}_t, (\nu_{p_\infty;t})_{t\in(-\infty,0]})$, where p_i are Ricci vertices with respect to a fixed reference $(1,1)$-form θ.

Theorem (Jian-Song-Tian 2023)

For almost every $t \in (-\infty, 0]$, the sequence $(M_i, d_{g_{i,t}}, p_i)$ converges in the pointed Gromov-Hausdorff sense to $(\mathcal{X}_t, p_t)$, and $\mathcal{X}_t$ has the structure of a normal analytic variety.

Applications: Fano fibrations have Type-I scalar curvature and diameter bounds at generic fibers. Projective bundles with Calabi ansatz undergoing a flip have Type-I curvature bounds.

Continuity of the Metric Flow

Theorem (H.-Jian-Song-Tian 2023)

$\mathcal{X}$ is a continuous metric flow in the sense of Bamler, and there are $p_t \in \mathcal{X}_t$ such that $t \mapsto (\mathcal{X}_t, d_t, p_t)$ is continuous in the pointed Gromov-Hausdorff topology.

This is mostly an application of the distortion estimates. This allows us to upgrade Jian-Song-Tian's structure theorem.

Corollary (H.-Jian-Song-Tian 2023)

The Gromov-Hausdorff convergence $(M_i, d_{g_{i,t}}, p_i) \to (\mathcal{X}_t, d_t, p_t)$ occurs at every time $t \in (-\infty, 0]$, and is locally uniform; moreover, every $\mathcal{X}_t$ is normal analytic variety.

Infinitesimal Structure

Theorem (H.-Jian-Song-Tian 2023)

Each $(\mathcal{X}_t, d_t)$ is a singular space with singularities of codimension ≥ 4. Any tangent cone of $(\mathcal{X}_t, d_t)$ is a metric cone.

Proof idea: The codimension 4 part follows from the estimates for the size of the almost-singular set.
For the tangent cone, Bamler's theory gives that any parabolic rescaling is a static cone; use the distortion estimates to identify with the tangent cone.

Remark (Improved description of tangent cones)

In future work, we will show that any tangent cone of $(\mathcal{X}_t, d_t)$ at x has the structure of an affine algebraic variety, uniquely and algebraically determined by the germ $(\mathcal{X}, x)$ using ideas of Donaldson-Sun (2017).

Future Directions

Conjecture (Jian-Song-Tian)

For each $t \in (-\infty, 0]$, $\mathcal{X}_t$ is quasi-projective.

Conjecture (Jian-Song-Tian)

For some choice(s) of reference $(1,1)$-forms θ, the Ricci vertices p_t and H_{2n}-centers z_t satisfy

$$d_t(p_t, z_t) \leq C\sqrt{1-t}.$$

This would imply any tangent flow in the sense of Bamler is a normal analytic variety.

Thank you for your attention.