Tangent Cones of Kähler-Ricci Flow Singularity Models

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Outline

Kähler-Einstein manifolds

Definition (Kähler Manifold)

A Kähler manifold (M, J, g) is a Riemannian manifold with an almost-complex structure satisfying $g(J\cdot, J\cdot) = g$ and $\nabla J = 0$.

Then $\omega := g(J\cdot, \cdot)$ is a symplectic form.

Example

$\mathbb{C}^n$, $\mathbb{C}P^n$, and any of their complex submanifolds.

If $Rc(g) = g$ for some $\lambda \in \mathbb{R}$, we say (M, g, J) is Fano Kähler-Einstein.

Metric structure

Suppose (M_i, g_i, J_i) is a sequence of Fano Kähler-Einstein manifolds with $\text{Vol}_{g_i}(M_i) \geq \nu > 0$.

Theorem (Metric convergence: Gromov, Cheeger-Colding-Naber (1981-2014))

A subsequence of (M_i, g_i) converge in the Gromov-Hausdorff sense to a metric space (X, d) such that:
(i) $X = \mathcal{R} \sqcup \mathcal{S}$, where $(\mathcal{R}, g, J)$ is a Kähler-Einstein manifold with metric completion (X, d),
(ii) $\dim_{\mathcal{H}}(\mathcal{S}) \leq 2n - 4$,
(iii) any pointed Gromov-Hausdorff limit $\lim_{i\to\infty}(X, \lambda_i d, p)$ with $\lambda_i \nearrow \infty$ is a Ricci-flat Kahler cone ($dr^2 + r^2 g_Z$ on the regular part).

Example: Asymptotic cone of the Eguchi-Hanson metric.
Fact: (Kodaira, Chow) Each (M_i, J_i) is a smooth projective variety.

Algebraic structure

Natural question: Given that (M_i, g_i, J_i) is a sequence of algebraic varieties, is the limit (X, d) also algebraic?

Theorem (Donaldson-Sun (2012, 2014))

(X, d) *is homeomorphic to a projective variety with log terminal singularities, whose regular set is* $\mathcal{R}$. *Also, metric cones of* X *are affine algebraic varieties, determined uniquely and algebraically.*

Theorem (Liu-Szekelyhidi (2018, 2019))

If instead of $Rc(g_i) = g_i$, *we have* $Rc(g_i) \geq -g_i$, *then tangent cones are still algebraic varieties.*

Important applications:

- Yau-Tian-Donaldson conjecture (Chen-Donaldson-Sun 2012)
- Yau's finite generation conjecture (Liu 2015)

Ricci Flow

Definition

A smooth, 1-parameter family of Riemannian metrics $(g_t)_{t\in[0,T)}$ on a closed manifold M^n satisfies Ricci flow if

$$\partial_t g_t = -2Rc(g_t),$$

where $Rc(g_t)$ is the Ricci curvature.

Fact: If g_0 is Kähler with complex structure J, then (M, g_t, J) is Kähler.

Shrinking Ricci Solitons

Definition

A shrinking gradient Ricci soliton (M^n, g, f) is a Riemannian manifold with $f \in C^\infty(M)$ satisfying

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

- If $\partial_t \varphi_t(x) = \frac{1}{1-t} \nabla f(\varphi_t(x))$, then

$$g(t) = (1-t)\varphi_t^* g$$

  solves the Ricci flow.
- Frequently model finite-time singularities of Ricci flow.
- Examples: Einstein manifolds, shrinking cylinder, FIK soliton.
- If (M, g, J) is Kähler, then $\mathcal{L}_{\nabla f} J = 0$.

Conjugate Heat Kernels

Definition (Conjugate heat kernel)

The conjugate heat equation is

$$\square^* u := (-\partial_t - \Delta + R)u = 0.$$

The conjugate heat kernel based at (x, t) is the function $(y, s) \mapsto K(x, t; y, s)$ satisfying

$$\square^*_{y,s} K(x, t; y, s) = 0$$
$$\lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x$$

Then $d\nu_{x,t;s} = K(x, t; \cdot, s) dg_s$ is a probability measure for each $s < t$.

Bamler's idea: Use the measures $\nu_{x,t;s}$ (instead of Riemannian volume measure) to describe convergence, and singular solutions.

Tangent Flows

Let $(M^n, (g_t)_{t\in[0,T)})$ be a closed Ricci flow with conjugate heat flow $\nu_{x,T;t} := \lim_{t_i \nearrow T} \nu_{x,t_i;t}$ based at the singular time.

Theorem (Bamler, 2020)

If $\tau_i \searrow 0$, $g_t^i := \tau_i^{-1} g_{T+\tau_i t}$, $\nu_t^i := \nu_{x,T;T+\tau_i t}$, *then we have* $\mathbb{F}$*-convergence*

$$(M, (g_t^i)_{t\in[-\tau_i T,0)}, (\nu_t^i)_{t\in[-\tau_i T,0)}) \to (\mathcal{X}, (\mu_t)_{t<0}),$$

where $\mathcal{X}$ *is a metric flow corresponding to a singular shrinking soliton* $(X, d, \mathcal{R}, g, f)$ *with singularities of codimension 4, and*

$$d\mu_t = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg_t.$$

Convergence is smooth on the regular set $\mathcal{R}$*.*

If $(M, (g_t)_{t\in[0,T)})$ is Kähler, then $(\mathcal{R}, g, f)$ is a Kähler-Ricci soliton. Also, $d_t|(\mathcal{R}_t \times \mathcal{R}_t)$ is the length metric of $(\mathcal{R}_t, d_{g_t})$. For any $x \in \mathcal{X}$, any tangent flow at x is a static flow modeled on a Ricci-flat cone.

Example of a tangent flow

Example (Feldman-Ilmanen-Knopf (2003))

There is a Kähler-Ricci soliton on $\mathcal{O}_{\mathbb{C}P^1}(-1)$ which converges to its asymptotic cone at $t = 0$.

Example (Song-Weinkove (2010), Song-Guo (2015))

If $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, with certain ω_0, then Ricci flow contracts an embedded $\mathbb{C}P^1$. Any tangent flow based at the contracted $\mathbb{C}P^1$ is the FIK soliton.

Previous work: Surfaces and Fano Manifolds

Theorem (Tian-Miao (2022))

Even in the Fano case, X can be singular when $n \geq 3$.

Theorem (Chen-Sun-Wang (2015), Dervan-Székelyhidi (2016), Han-Li (2020))

If $\omega_0 \in c_1(M)$, then X is a projective variety with log terminal singularities, uniquely and algebraically determined by (M, J).

This was used in Chen-Sun-Wang (2015) to give a Ricci flow proof of the YTD conjecture.

Theorem (Bamler (2020))

Any tangent flow of surfaces has only discrete orbifold singularities, with tangent cone $\mathbb{C}^2/\Gamma$ for some $\Gamma \leq U(2)$.

Local Model: Soliton Transition

Theorem (Song-Tian (2009))

The Kähler-Ricci flow starting on a projective manifold can be continued through singularities, corresponding to the minimal model program.

Conjecture (Song (2013))

A parabolic rescaling at any exceptional point of a birational surgery performed by Ricci flow on projective varieties converges to a shrinking-expanding soliton transition.

Example (Chi Li's Solitons (2010))

There is a Kähler-Ricci soliton on $\mathcal{O}_{\mathbb{C}P^m}(-1)^{(n+1)}$ which converges to its asymptotic cone at $t = 0$, and then flows into a soliton on $\mathcal{O}_{\mathbb{C}P^n}(-1)^{(m+1)}$ if $n < m$.

Singularity Models of Projective KRF

Suppose $(M_i, (g_{i,t})_{t\in[-T_i,0]}, (\nu_{p_i,0;t})_{t\in[-T_i,0]})$ is a sequence of Type-I rescalings of a projective Kähler-Ricci flow $\mathbb{F}$-converging to a future-continuous metric flow $(\mathcal{X}_t, (\nu_{p_\infty;t})_{t\in(-\infty,0]})$, where p_i are Ricci vertices with respect to a fixed reference $(1,1)$-form θ.

Theorem (Jian-Song-Tian 2023)

For almost every $t \in (-\infty, 0]$, the sequence $(M_i, d_{g_{i,t}}, p_i)$ converges in the pointed Gromov-Hausdorff sense to $(\mathcal{X}_t, p_t)$, and $\mathcal{X}_t$ has the structure of a normal analytic variety.

Applications: Fano fibrations have Type-I scalar curvature and diameter bounds at generic fibers. Projective bundles with Calabi ansatz undergoing a flip have Chi Li's solitons as their tangent flows.

Theorem (H.-Jian-Song-Tian 2023)

The Gromov-Hausdorff convergence $(M_i, d_{g_{i,t}}, p_i) \to (\mathcal{X}_t, d_t, p_t)$ occurs at every time $t \in (-\infty, 0]$; moreover, the flow is continuous, and every $\mathcal{X}_t$ is normal analytic variety, whose tangent cones are Ricci-flat metric cones.

Main theorem

Suppose X is any tangent flow of a compact Kähler-Ricci flow.

Theorem (H. (2023))

Any tangent cone of X is homeomorphic to a normal affine algebraic variety, and the metric and algebraic singular sets agree.

In fact, the same is true for any limit of compact noncollapsed Kähler-Ricci flows which is a Ricci-flat cone (e.g. asymptotic cones, tangent cones of Ricci-flat singularity models).

Proof strategy

To construct "enough" holomorphic functions, the strategy is standard.
Model case: (Ricci-flat cone) Then the trivial line bundle with the constant section, and Hermitian metric $e^{-\frac{d^2(o,\cdot)}{4}}$ is a "peak" section.
General case:

1. The limit space is infinitesimally conical, with small singular set, so the "peak" section from a tangent cone at $x \in X$ can be grafted to get an almost-holomorphic "peak" section near x.
2. Solve a $\overline{\partial}$-equation on the singular space (or smooth approximants) to perturb to a holomorphic "peak" section.

For Step 1, use Bamler's compactness theory.

Hörmander estimate on tangent flows

To solve the $\overline{\partial}$-equation on X (with estimates), it suffices to prove a weighted L^2 estimate.

Proposition (H. (2023))

If X is a tangent flow, and $v \in C_c^\infty(\mathcal{R}_X, \mathbb{C})$, then for all $\eta \in \mathcal{A}_c^{0,1}(\mathcal{R}_X)$,

$$\left| \int_{\mathcal{R}_X} \langle \eta, \overline{\overline{\partial} v} \rangle d\nu \right| \leq \left(\int_{\mathcal{R}_X} |\overline{\partial}_f^* \eta|^2 d\nu \right)^{\frac{1}{2}} \left(\int_{\mathcal{R}_X} |\overline{\partial} v|^2 d\nu \right)^{\frac{1}{2}}.$$

If σ is an almost-holomorphic section, then Hilbert space theory and the L^2 estimate give a correction u such that $\overline{\partial} u = \overline{\partial} \sigma$, and u is small. Combine with elliptic estimates.

A parabolic approach to Hörmander estimates

For simplicity, assume X is a smooth and complete KRS. Fix $t_0 < t_1 < 0$, and let $(v_t d\nu_t)_{t<t_1}$ be the conjugate heat flow ending at $v d\nu_{t_1}$.

Proposition (H. (2023))

For all $\eta \in \mathcal{A}_c^{0,1}(X)$,

$$\left| \int_X \langle \eta, \overline{\overline{\partial} v_{t_0}} \rangle d\nu_{t_0} \right| \leq \left(\int_X |\overline{\partial}_f^* \eta|^2 d\nu_{t_0} \right)^{\frac{1}{2}} \left(|t_1| \int_X |\overline{\partial} v|^2 d\nu_{t_1} \right)^{\frac{1}{2}}$$

The claim will then follow by taking $t_0 \nearrow t_1$.
To prove this, write $\eta = \eta^{(1)} + \eta^{(2)}$, where $\eta^{(1)} \in \ker(\overline{\partial})$, $\eta^{(2)} \perp \ker(\overline{\partial})$.
Let $\eta_t = \eta_t^{(1)} + \eta_t^{(2)}$ solve

$$\partial_t \eta_t^{(j)} = -\Delta_{\overline{\partial}} \eta_t^{(j)}$$
$$\eta_{t_0}^{(j)} = \eta^{(j)}.$$

Proof in the model case

1. $t \mapsto |t| \int_X \langle \eta_t^{(j)}, \overline{\overline{\partial} v_t} \rangle d\nu_t$ is constant, and zero if $j = 2$, so

$$|t_0| \left| \int_X \langle \eta, \overline{\overline{\partial} v_{t_0}} \rangle d\nu_{t_0} \right| \leq |t_1| \left(\int_X |\eta_{t_1}^{(1)}|^2 d\nu_{t_1} \right)^{\frac{1}{2}} \left(\int_X |\overline{\partial} v|^2 d\nu_{t_1} \right)^{\frac{1}{2}},$$

2. $(\partial_t - \Delta)|\eta_t^{(j)}| \leq 0$, so $|\eta_t^{(j)}|$ is locally bounded for $t > t_0$,
3. By (2), we can integrate the Kodaira-Nakano formula by parts, and use the Ricci soliton equation to get

$$\int_X |\eta_{t_1}^{(1)}|^2 d\nu_{t_1} \leq |t_1| \int_X |\overline{\partial}_f^* \eta_{t_1}|^2 d\nu_{t_1},$$

4. $(\partial_t - \Delta)|t \overline{\partial}_f^* \eta_t| \leq 0$, so

$$|t_1|^2 \int_X |\overline{\partial}_f^* \eta_{t_1}|^2 d\nu_{t_1} \leq |t_0|^2 \int_X |\overline{\partial}_f^* \eta_{t_0}|^2 d\nu_{t_0}.$$

The general case

Some ingredients:

- Instead of working on a singular soliton, work on a smooth almost-soliton
- View conjugate heat kernel measure $d\nu_t$ as an "almost-polarization"
- Conjugate heat kernel estimates (hypercontractivity, ultracontractivity) by Bamler (2020)
- Replace f with parabolic regularization h constructed in H.-Jian. (2022)
- Crucial inequality:

$$(\partial_t - \Delta)\left(-aw + \frac{1}{a}|\eta|^2 + 2|t\overline{\partial}_h^*\eta|\right) \leq 0$$

where w is a heat equation version of Perelman's Harnack quantity, and $a \gg 0$

The regular set

To show that algebraically regular points $x \in X$ are analytically regular, modify an argument of Szekelyhidi-Liu:

- Choose $h_1, ..., h_n : X \to \mathbb{C}$ giving a biholomorphism from a neighborhood of x
- Using Baldauf's parabolic frequency for Ricci flow, show that either $(h_1, ..., h_n)$ is an almost-splitting map (in which case we are done), or one of h_i has faster-than-linear decay
- Iterate the argument, if one of the h_i always shrinks, then $\log |dh_1 \wedge \cdots \wedge dh_n|$ has positive Lelong number
- $\log |dh_1 \wedge \cdots \wedge dh_n|$ is pluriharmonic outside a set of codimension 4, so has zero Lelong number, a contradiction

Future directions

- Do tangent cones always have log-terminal singularities?
- When are tangent flows complex spaces? Quasi-projective?
- Do Kähler-Ricci flows always admit finite-depth "bubble trees"?