Entropy Convergence of Ricci Flows with a Type-I Scalar Curvature Bound

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Overview

Ricci Flow

Notation

(M^n, g) is an n-dimensional Riemannian manifold.

- Rm is the Riemannian curvature tensor,
- Rc is the Ricci curvature,
- R is the scalar curvature.

Ricci Flow

Introduction

Problem : Given a smooth manifold M, find a "canonical Riemannian metric" g on M.

- M admits g with constant sectional curvature $\iff$ M is a quotient of $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$.
- If $n \geq 3$, M admits a metric g with constant scalar curvature in every conformal class (Yamabe problem).
- The "Constant Ricci curvature" condition is the Einstein equation : $Rc(g) = \lambda g$.

PDE Perspective : There are $n(n+1)/2$ metric components.

- $Rm = cI$ is overdetermined : ($\frac{1}{12}n^2(n-1)(n+1)$ equations),
- $R = c$ is underdetermined : (1 equation).

Ricci Flow

Definition

Richard Hamilton's idea : "Heat flow" to an Einstein metric

Definition (Ricci Flow)

A solution of (normalized) Ricci flow is a smooth family of metrics $(M^n, g(t)_{t\in[0,T)})$ *satisfying*

$$\frac{\partial g(t)}{\partial t} = \lambda g(t) - 2Rc(g(t)).$$

The linearized operator DRc_g is not strictly elliptic :
for a 1-parameter family (φ_t) with $\partial_t|_{t=0}\varphi_t = X^{\#}$,

$$DRc_g(\delta_g^*(X)) = DRc_g(\partial_t \varphi_t^* g)|_{t=0} = \partial_t \varphi_t^* Rc(g)|_{t=0} = \mathcal{L}_{X^\#} Rc(g),$$

where $\delta_g^*(X) = \mathcal{L}_{X^\#} g = \mathrm{Sym}(\nabla X)$ is the symmetrized covariant derivative. $DRc_g \circ \delta_g^*$ **is only first order in** X **!**

Ricci Flow

Ricci Flow as a Parabolic Equation

- For a background metric g_0, there is a g-dependent vector field $X(g, g_0)$ such that the Ricci-deTurck operator

$$P(g, g_0) = -2Rc(g) + \mathcal{L}_{X(g,g_0)} g$$

  is elliptic in g. Then Ricci-deTurck flow $\partial_t g(t) = P(g(t), g_0)$ has a unique solution for small time, and $\tilde{g}(t) = \varphi_t^* g(t)$ solves Ricci flow, where (φ_t) is the flow of X.
- Ricci flow can be considered as a dynamical system in the space of Riemannian metrics on M, modulo scaling and diffeomorphisms.

Ricci Flow

Fixed Points

- Up to scaling $\sigma(t)$ and diffeomorphisms (φ_t), "fixed points" of Ricci flow are self-similar solutions :

$$g(t) = \sigma(t)\varphi_t^* g.$$

- Such solutions are equivalent to a fixed metric g and a vector field X satisfying the Ricci soliton equation :

$$Rc(g) + \mathcal{L}_X g = \lambda g.$$

- If $X = \nabla f$, then (M, g, f) is a gradient Ricci soliton.
- Different properties when $\lambda < 0, \lambda = 0, \lambda > 0$ (expanding, steady, shrinking).
- Solitons will occur as dilation limits of Ricci flow.

Ricci Flow

Applications

Some geometric/topological applications of Ricci flow :

- 3-dimensional Riemannian manifolds with positive Ricci curvature are space forms (Hamilton 1982)
- Simply connected compact 3-manifolds are diffeomorphic to $\mathbb{S}^3$ (Perelman 2002)
- Kahler-Einstein metrics exist on K-stable Fano manifolds (X. Chen, B. Wang, S. Sun 2012, Tian 2012)

Common technique : Understand how singularities form.

Ricci Flow

Finite-Time Singularities

Theorem (Hamilton 1982)

If a Ricci flow $(M, g(t)_{t\in[0,T)})$ *cannot be extended past time* $T < \infty$*, then*

$$\liminf_{t\to T} \sup_{x\in M} |Rm|(x,t)(T-t) > 0.$$

Theorem (X. Chen, B. Wang 2011)

$$\limsup_{t\to T} \sup_{x\in M} |Rc|(x,t)(T-t) > 0.$$

Open Problem : Is it possible for $\sup_{M\times[0,T)} |R|(x,t) < \infty$?

Geometric Convergence

Smooth Convergence, and Compactness

Definition (Cheeger-Gromov (smooth) Convergence)

A sequence (M_i, g_i, p_i) of pointed Riemannian manifolds converges to $(M_\infty, g_\infty, p_\infty)$ in the Cheeger-Gromov sense if there exist diffeomorphisms $\varphi_i : U_i \to M_i$ with $\cup_i U_i = M_\infty$ and

$$\sup_K |\nabla^k_{g_\infty}(\varphi_i^* g_i - g_\infty)| \to 0$$

Theorem (Cheeger 1970)

If (M_i^n, g_i, p_i) is a sequence of complete pointed Riemannian manifolds with $|\nabla^k_{g_i} Rm(g_i)|_{g_i} \leq C_k$ and $Vol_{g_i}(B(p_i, 1)) > \nu > 0$, then a subsequence converges in the Cheeger-Gromov sense to a complete Riemannian manifold.

Ricci Flow

Type-I Solutions

Idea : Try to find limits of $(M, \lambda_i g(t_i), p_i)$, where $\lambda_i \to \infty$ and $|Rm|(p_i, t_i) \to \infty$.
Difficulties : In general, such limits may not exist, or may be difficult to classify.

Definition (Type-I Solutions of Ricci Flow)

A solution $g(t)_{t\in[0,T)}$ of Ricci flow is Type-I if

$$\limsup_{t\to T} \sup_{x\in M} |Rm|(x,t)(T-t) < \infty.$$

Examples : neckpinch, shrinking solitons

Ricci Flow

Type-I Singularities

Theorem (Naber 2007, Enders-Muller-Topping 2010, Mantegazza-Muller 2012, X. Cao-Q. Zhang 2010)

At any point $p \in M$ of a Type-I Ricci flow, there is a sequence $t_i \nearrow T$ such that $(M, (T-t_i)^{-1} g(t_i), p)$ converges in the pointed Cheeger-Gromov sense to a gradient shrinking soliton.

- *The singular set*

$$\Sigma := \{q \in M;\ \sup_{U \times [0,T)} |Rm| = \infty \text{ for every neighborhood } U \text{ of } q\}$$

  is the set of $q \in M$ such that $\lim_{t \to T} |R|(x,t)(T-t) < \infty$.
- *If $p \in \Sigma$, then the soliton is nonflat.*
- *Σ is closed and nonempty.*

Model Spaces

Nonsmooth Limit Spaces

Natural Question : How much can we generalize this description of singularities ?

- In general, we have to describe Ricci flow solutions where curvature is large, but possibly smaller than $\sup_M |Rm|(\cdot, t)$ (e.g. Perelman's canonical neighborhood theorem).
- One approach is to consider possibly nonsmooth model spaces.

Type-I Scalar Curvature Bounds

Definition and Examples

Definition (Type-I Scalar Curvature Bounds)

A Ricci flow $(M, g(t)_{t\in[0,T)})$ has Type-I scalar curvature if :

$$\sup_{t\in[0,T)} \sup_{x\in M} |R|(x,t)(T-t) < \infty.$$

- Equivalently, the rescaled flow $\tilde{g}_t := e^t g_{T-e^{-t}}$, which solves

$$\partial_t \tilde{g}_t = \tilde{g}_t - 2Rc(\tilde{g}_t),$$

  has bounded scalar curvature.
- Examples include Type-I Ricci flows, and Kahler-Ricci flows on Fano manifolds (Type-II Ricci flows on G-compactifications, Li-Tian-Zhu 2020)

Model Spaces

Ricci Flows with Bounded Scalar Curvature

- The scalar curvature evolves under Ricci flow by

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2$$

- If scalar curvature is bounded, any smooth blowup limit has zero scalar curvature, so is Ricci flat (actually ALE, so can be ruled out by topological assumptions).
- Limits are "infinitesimally Ricci flat"

Metric Convergence and Compactness

Definition (Gromov-Hausdorff (metric) Convergence)

A sequence (X_i, d_i, p_i) *of pointed metric spaces converges to* (X, d, p) *in the Gromov-Hausdorff sense if there exist* $\psi_i : B(p, i) \to X_i$ *with* $|d_X(x, y) - d_{X_i}(\psi_i(x), \psi_i(y))| < 1/i$, $d_{X_i}(p_i, \psi_i(p)) < 1/i$, *that are* $(1/i)$*-dense in* $B(p_i, i)$.

Theorem (Anderson, Cheeger, Colding, Gromov, Naber, Tian)

If (M_i^n, g_i, p_i) *is a sequence of pointed Riemannian manifolds with* $|Rc(g_i)| \leq C$ *and* $Vol_{g_i}(B(p_i, 1)) > \nu > 0$, *then a subsequence converges in the Gromov-Hausdorff sense to a complete metric length space, which is a* $C^{1,\alpha}$ *Riemannian manifold outside of a subset of Minkowski codimension 4.*

Model Spaces

Singular Model Spaces

Definition (Singular Space (Bamler 2016))

A singular space is a tuple $\mathcal{X} = (X, d, \mathcal{R}, g)$, where (X, d) is a complete metric length space, and $(\mathcal{R}, g)$ is a dense open subset with the structure of a C^∞ Riemannian manifold, such that the induced length metric is d.

- Also require local upper and lower volume bounds.
- $\mathcal{X}$ has singularities of codimension k if $X \setminus \mathcal{R}$ has Minkowski codimension at least k.
- $\mathcal{X}$ is Y-regular if almost-Euclidean volume implies a local bound on curvature.
- Convergence to a singular space means metric convergence everywhere, smooth convergence on $\mathcal{R}$.

Type-I Scalar Curvature Bounds

Bounded Scalar Curvature

Theorem (Bamler 2016)

If $(M_i^n, (g_t^i)_{t\in[0,2]}, q_i)$ are closed solutions of Ricci flow with $\mu[g_0^i, 4] \geq -A$ and $|R_{g_i}| \leq A$, then a subsequence converges to a regular singular space with singularities of codimension 4.

Bamler also showed that, for any $p \in [1,2)$, any time slice satisfies, for $(x,t) \in M \times [-1,0]$,

$$\int_{B(x,t,1)} |Rm|_{g_t}^p d\mu_{g_t} \leq C(A,p).$$

The set of large curvature points is uniformly small!

Type-I Scalar Curvature Bounds

Singular Shrinking Solitons

Theorem (Bamler 2016)

If $(M^n, (g_t)_{t\in[0,T)})$ has Type-I scalar curvature, and $q \in M$, then there exists $t_i \nearrow T$ such that $(M, (T-t_i)^{-1}g_{t_i}, q)$ converges to a singular shrinking gradient Ricci soliton, that is Y-regular.

Important application : Proof of the Yau-Tian-Donaldson conjecture
Weakness : Unlike the Kahler setting, (studied by B. Wang, X. Chen) flow convergence not available, no long-time pseudolocality

Perelman's Entropy Functional

The main tool for showing blowup limits are shrinking GRS is Perelman's entropy functional.

$$\mathcal{W}(g, f, \tau) = \int_M \left(\tau(R + |\nabla f|^2) + f - n\right) \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu_g.$$

Theorem (Entropy Monotonicity (Perelman 2002))

If $(M^n, g(t)_{t\in[0,T)})$ is a Ricci flow solution, $\tau(t) = T' - t$, and $(4\pi\tau(t))^{-\frac{n}{2}} e^{-f(t)}$ solves the conjugate heat equation, then (for $t < T'$)

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau(t) \int_M \left| Rc(g(\tau)) + \nabla^2 f(t) - \frac{g(t)}{2\tau(t)} \right|^2 \frac{e^{-f(t)}}{(4\pi\tau(t))^{\frac{n}{2}}} d\mu_{g(t)}$$

Soliton Entropy

Definition

The entropy of a singular shrinking GRS (M, g, f) is

$$\mathcal{W}(g, f) := \mathcal{W}(g, f, 1) = (4\pi)^{-\frac{n}{2}} \int_M (R + |\nabla f|^2 + f - n) e^{-f} d\mu_g.$$

A shrinking GRS is normalized if $(4\pi)^{-\frac{n}{2}} \int_M e^{-f} d\mu_g = 1$.

- The integral converges by quadratic bounds for f, volume growth estimates for (M, g)
- Entropy, normalization are preserved under the canonical Ricci flow
- If $(M, g, f_1), (M, g, f_2)$ are normalized shrinking GRS, then $\mathcal{W}(g, f_1) = \mathcal{W}(g, f_2)$ (Naber 2011)

Conjugate Heat Kernels at the Singular Time

The conjugate heat operator is

$$-\partial_t - \Delta + R_{g(t)}.$$

For any $(x,t) \in M \times (-2,0)$, there is a unique conjugate heat kernel $u_{x,t}$ based at (x,t). If $(x_j,t_j) \to (x,0)$, a limit $u_{x,0}$ is a conjugate heat kernel at the singular time.

- $u_{x,0}$ could possibly not be unique.
- Define $\theta_x(t) := \mathcal{W}(g(t), f_t(t), |t|)$, where $u_{x,0}(t) = (4\pi|t|)^{-\frac{n}{2}} e^{-f_t(t)}$ minimizes entropy among conjugate heat kernels at the singular time.
- (Mantegazza-Muller) $\Theta(x) = \lim_{t\to 0} \theta_x(t) = \mathcal{W}(g_\infty, f_\infty, 1)$, so entropy uniqueness for blowups.
- (Mantegazza-Muller) $\Sigma = \{x \in M \,;\, \Theta(x) < 0\}$.

Type-I Scalar Curvature Bounds

Entropy Convergence

Theorem (H-)

If $(M^n, (g(t))_{t\in[-2,0)}, q)$ satisfies a Type-I scalar curvature bound, and $(X, d, \mathcal{R}, g)$ is a limit of $(M, |t_i|^{-1}g(t_i), q)$, then :
i. $\Theta(q) = \lim_{t\to 0}\theta_q(t) = \mathcal{W}(g, f)$,
ii. $(4\pi)^{-\frac{n}{2}}\int_{\mathcal{R}} e^{-f}d\mu_g = 1$,
iii. $\mathcal{W}(g, f)$ *only depends on* g,
iv. $\Sigma = \{x \in M \,;\, \Theta(x) < 0\}$.
In particular, entropy uniqueness for Type-I blowups at a fixed point.

Summary : The Gaussian density and soliton entropy have the same properties as in the Type-I curvature setting.

Proof of Entropy, Heat Content Convergence

Main idea : Gaussian-type heat kernel estimates and Bamler's L^p curvature estimates imply the integrand of $\mathcal{W}$ does not escape to infinity, or concentrate on the singular set.

- Gaussian-type heat kernel estimates for the conjugate heat kernel imply

$$-C + C^{-1}d^2(x_0,\cdot) \leq f \leq C + Cd^2(x_0,\cdot).$$

- Integrate by parts :

$$\mathcal{W}(g,f) = \int_M \left(\tau(R + 2\Delta f - |\nabla f|^2) + f - n\right)(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu_g.$$

Soliton Entropy only Depends on Riemannian Metric

Proof in the smooth setting : If $f_1, f_2 \in C^\infty(M)$ are distinct potential functions, then $\nabla^2(f_1 - f_2) = 0$, so $\nabla(f_1 - f_2)$ induces a metric splitting. Then use normalization condition.
Difficulty : $\nabla(f_1 - f_2)$ might not be complete.
Strategy : Singularities are of codimension 4, so an argument of B. Wang, X. Chen shows that the flow exists for all time outside a codimension 3 set, which gives a metric $\mathcal{R} \cong N \times \mathbb{R}$ splitting outside a small subset $N \subseteq \{f_1 = f_2\}$. Use normalization condition (the argument is more subtle since N might not be connected).

Characterization of the Singular Set

Proof in the Smooth Setting : Contradiction-compactness argument, entropy rigidity of Euclidean space (Yokota).
Difficulty : The entropy rigidity of Euclidean space relies on completeness.
Strategy : Argue on the original flow $(M_i, |t_i|^{-1}g(t_i), q)$. If $x \in \mathcal{R}$, and $\phi_i : U_i \to M_i$ are open embeddings realizing smooth convergence, then $(\phi_i(x), t_i) \to (q, 0)$, so $u_{\phi_i(x),t_i}$ converge to a conjugate heat kernel $u = (4\pi|t|)^{-\frac{n}{2}}e^{-f}$ at the singular time with

$$\lim_{t \nearrow 0} \mathcal{W}(g(t), f(t), \tau(t)) = 0.$$

Thus $\limsup_{i\to\infty} \mathcal{W}_{\phi_i(x),t_i}(\delta) > \epsilon_0$, so apply the Hein-Naber ϵ-regularity theorem.

Additional Structure in Dimension 4

Theorem (H-)

If $(M^4, (g_t)_{t\in[0,T)}, p)$ is a closed Ricci flow with a Type-I scalar curvature bound, then the limiting singular GRS $(X, d, \mathcal{R}, g)$ has the structure of a C^∞ Riemannian orbifold with finitely many isolated conical singularities, satisfying the Ricci soliton equation everywhere.

Any $x \in X \setminus \mathcal{R}$ has a neighborhood U, quotient map

$$\mathbb{R}^n \supseteq B \xrightarrow{\pi} B/\Gamma \cong U \subseteq X$$

for some finite subgroup $\Gamma \leq O(4, \mathbb{R})$, such that $\pi^* g$, $\pi^* f$ extend smoothly to B.

Removal of Singularities Technique

- Show that singularities are isolated, and $|Rm| \leq o(r^{-2})$
- Show that, for any $x \in X \setminus \mathcal{R}$, $\int_{B^*(x,r)} |Rm|^2 d\mu_g < \infty$
- Write $Rm = dA + [A, A]$, where A are the Christoffel symbols with respect to a broken Hodge gauge : $(d^* A = 0)$.
- For Ricci solitons, use the Yang-Mills type equation

$$\nabla^* Rm = Rm(\nabla f).$$

- Using ϵ-regularity (S. Huang 2017), conclude $|Rm|(x) \leq Cd^{-\delta}(x, x_0)$ for some $\delta \in (0, 1)$, splice almost-linear coordinates.

Singularities are Isolated

Difficulty : A set with Minkowski dimension 0 need not be discrete.
Strategy : A limit point of $X \setminus \mathcal{R}$ would have a tangent cone with singular points outside of the vertex, but tangent cones are metric cones.

Difficulty : For this to work, a sequence of balls in X centered at singular points cannot converge in the Gromov-Hausdorff sense to a Euclidean ball.
Strategy : The L^p curvature bounds for $(M_i, |t_i|^{-1}g(t_i))$ imply L^{2p} Ricci curvature bounds, $2p > \frac{n}{2}$ when $n < 8$. By noncollapsing, apply variant of Cheeger-Colding theory.

Local L^2 Curvature Estimate

Difficulties :

- Even a Type-I Ricci flow can have unbounded diameter after Type-I rescaling.
- Bamler-Zhang's bound on $||Rm||_{L^2}$ deteriorates if the diameter is unbounded.

Thus we are forced to argue directly on the limit space.
Strategy : Decompose curvature into scalar, trace-free Ricci, and Weyl tensors, and estimate each piece directly.

- Scalar curvature is bounded by assumption
- To estimate trace-free Ricci, use Haslhoffer-Muller's argument
- Estimate self-dual, anti-self-dual parts of the Weyl tensor seperately, using Chern-Simons invariants

Related Problems

- Are conjugate heat kernels at the singular time determined by their basepoint in the Type-I setting ? In the Type-I scalar curvature setting ?
- Are Type-I blowups determined by their basepoint ?
- Are there distinct normalized shrinking GRS with the same entropy ?
- Find more examples of Type-II Ricci flows with Type-I scalar curvature (Stolarski doubly warped product ?)