Tangent Flows of Kähler Metric Flows

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Outline

1. Ricci Flows and Metric Flows
2. Stratification of Singular Sets
3. Improved Splitting for Kähler Metric Flows
4. Isometric Actions on Singular Solitons

Ricci Flow

Definition

A smooth, 1-parameter family of Riemannian metrics $(g_t)_{t\in[0,T)}$ on a closed manifold M^n satisfies Ricci flow if

$$\partial_t g_t = -2Rc(g_t),$$

where $Rc(g_t)$ is the Ricci curvature.

Ricci flow was first used by Richard Hamilton in 1982, to prove that every 3-dim Riemannian manifold with positive Ricci curvature is a space form.

Basic Facts About Ricci Flow

- (PDE Classification) Ricci flow is a second-order nonlinear weakly parabolic system. In harmonic coordinates,

$$\partial_t g_{t,ij} = -2Rc(g_t)_{ij} = \Delta g_{t,ij} + Q_{ij}(g_t, Dg_t).$$

- (Short-time existence/uniqueness) For any smooth Riemannian metric g on M, there is a unique solution $(M^n, (g_t)_{t\in[0,T)})$ of Ricci flow with $g_0 = g$.
- (Holonomy is Preserved) If g_0 is Kähler with complex structure J, then (M, g_t, J) is Kähler.
  (Recall (M, g, J) is Kähler if $J^2 = -I$, $g(JX, JY) = g(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$, and $\nabla J = 0$)

Shrinking Ricci Solitons

Definition

A shrinking gradient Ricci soliton (M^n, g, f) is a Riemannian manifold with $f \in C^\infty(M)$ satisfying

$$Rc + \nabla^2 f = \frac{1}{2} g.$$

- If $\partial_t \varphi_t(x) = \frac{1}{1-t} \nabla f(\varphi_t(x))$, then

  $$g(t) = (1-t)\varphi_t^* g$$

  solves the Ricci flow.
- Frequently model finite-time singularities of Ricci flow.
- Examples: Einstein manifolds, shrinking cylinder, FIK soliton.
- If (M, g, J) is Kähler, then $\mathcal{L}_{\nabla f} J = 0$.

Conjugate Heat Kernels

Definition (Conjugate heat kernel)

The conjugate heat equation is

$$\Box^* u := (-\partial_t - \Delta + R)u = 0.$$

The conjugate heat kernel based at (x, t) is the function $(y, s) \mapsto K(x, t; y, s)$ satisfying $\lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x$ and $\Box^*_{y,s} K(x, t; y, s) = 0$. Then $d\nu_{x,t;s} = K(x, t; \cdot, s)dg_s$ is a probability measure for each $s < t$.

Definition

The pointed Nash entropy based at (x, t) at the scale r is

$$\mathcal{N}_{x,t}(r^2) := \int_M f_{x,t}(\cdot, t - r^2)d\nu_{x,t;t-r^2} - \frac{n}{2},$$

where $K(x, t; y, s) = (4\pi(t - s))^{-\frac{n}{2}} e^{-f_{x,t}(y,s)}$

Metric Flows

Definition (Bamler 2020)

A metric flow is a set $\mathcal{X}$ with a time function $\mathfrak{t} : \mathcal{X} \to \mathbb{R}$ whose time slices $\mathcal{X}_t := \mathfrak{t}^{-1}(t)$ are equipped with metrics d_t, and for $x \in \mathcal{X}_t$, $s < t$, there are probability measures $\nu_{x,t;s}$ on $\mathcal{X}_s$ such that for $t_1 < t_2 < t_3$, $x \in \mathcal{X}_{t_3}$, and $A \subseteq \mathcal{X}_{t_1}$, we have

$$\nu_{x;t_1}(A) = \int_{\mathcal{X}_{t_2}} \nu_{y;t_2}(A) d\nu_{x;t_2}(y).$$

Smooth Case: $\mathcal{X} = M \times I$, $\mathfrak{t}$ the projection, $d_t = d_{g_t}$,

$$d\nu_{x,t;s} = K(x,t;\cdot,s)dg_s.$$

Definition (Kleiner-Lott 2014)

A Ricci flow spacetime $(\mathcal{R}, \mathfrak{t}, g, \partial_{\mathfrak{t}})$ is an $(n+1)$-manifold $\mathcal{R}$, a smooth function $\mathfrak{t}$, a vector field $\partial_{\mathfrak{t}}$ on $\mathcal{R}$ with $\partial_{\mathfrak{t}}\mathfrak{t} = 1$ and

$$\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2Rc(g).$$

$\mathbb{F}$-Convergence

Definition (Wasserstein Distance)

For measures μ, ν on a metric space (X, d), the Wasserstein distance is

$$d_{W_1}(\mu, \nu) := \inf_{\pi} \int_{X \times X} d(x, y) d\pi(x, y),$$

where the infimum is taken over all couplings π of (μ, ν).

Gromov-Wasserstein distance: When measures do not live on the same metric space, take infimum of the Wasserstein distance over all isometric embeddings into a common metric space.

$\mathbb{F}$**-Convergence:** A sequence $(M_i, (g_t^i)_{t \in I^i}, (\nu_t^i)_{t \in I^i})$ of Ricci flows with reference conjugate heat flows $\mathbb{F}$**-converges** to a metric flow $(\mathcal{X}, (\nu_t^\infty)_{t \in I^\infty})$ if the time slices converge in the Gromov-Wasserstein distance at almost-every time.

Tangent Flows

Let $(M^n, (g_t)_{t\in[0,T)})$ be a closed Ricci flow with conjugate heat flow $\nu_{x,T;t} := \lim_{t_i \nearrow T} \nu_{x,t_i;t}$ based at the singular time.

Theorem (Bamler, 2020)

If $\tau_i \searrow 0$, $g_t^i := \tau_i^{-1} g_{T+\tau_i t}$, $\nu_t^i := \nu_{x,T;T+\tau_i t}$, *then we have* $\mathbb{F}$*-convergence*

$$(M, (g_t^i)_{t\in[-\tau_i T,0)}, (\nu_t^i)_{t\in[-\tau_i T,0)}) \to (\mathcal{X}, (\mu_t)_{t<0}),$$

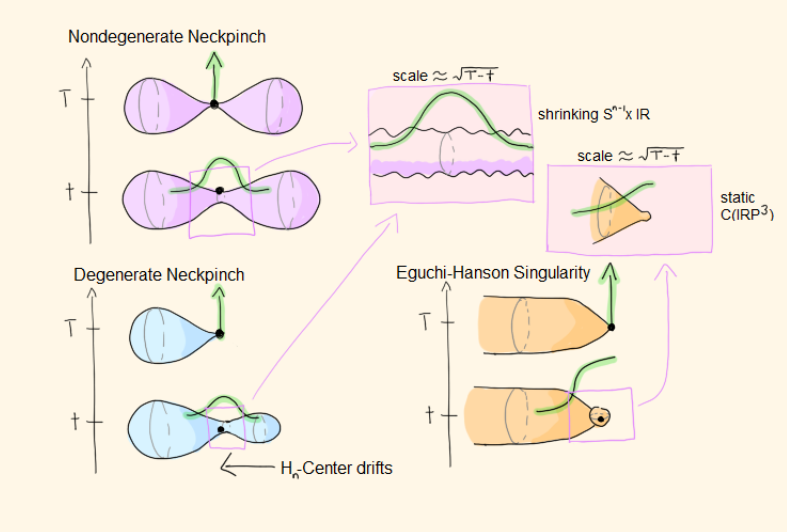
where $\mathcal{X}$ *is a metric flow corresponding to a singular shrinking soliton* $(X, d, \mathcal{R}, g, f)$ *with singularities of codimension 4, and*

$$d\mu_t = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg_t.$$

Convergence is smooth on the regular set $\mathcal{R}$*.*

If $(M, (g_t)_{t\in[0,T)})$ is Kähler, then $(\mathcal{R}, g, f)$ is a Kähler-Ricci soliton. Also, $d_t|(\mathcal{R}_t \times \mathcal{R}_t)$ is the length metric of $(\mathcal{R}_t, d_{g_t})$. For any $x \in \mathcal{X}$, any tangent flow at x is a singular shrinking Ricci soliton.

Tangent Flows



Structure Theory of Metric Flows

Theorem (Bamler 2020)

Given Ricci flows $(M_j, (g_{j,t})_{t\in[-T_j,0]}, (\nu_{x_j,0;t})_{t\in[-T_j,0]})$ with $\mathcal{N}_{x_j,0}(1) \geq -Y$, a subsequence $\mathbb{F}$-converges to some metric flow pair $(\mathcal{X}, (\nu_{x_\infty;t})_{t\in[-T_\infty,0]})$ over $[-T_\infty, 0]$. Also, $\mathcal{X} = \mathcal{R} \sqcup \mathcal{S}$, where $\mathcal{R}$ has the structure of a smooth Ricci flow spacetime, and $\mathcal{S}$ has P^*-parabolic dimension $\leq (n-2)$.

The tangent flow of any point $x \in \mathcal{X}$ is a singular shrinking soliton, and there is a stratification

$$\mathcal{S}^0 \subseteq \mathcal{S}^1 \subseteq \cdots \subseteq \mathcal{S}^{(n+2)-4} = \mathcal{S},$$

where for any point $x \in \mathcal{X} \setminus \mathcal{S}^k$, some tangent flow $\mathcal{X}'$ based at x satisfies one of the following:

(i) $\mathcal{X}' \cong \mathcal{X}'' \times \mathbb{R}^{k+1}$ for some singular soliton $\mathcal{X}''$,

(ii) $\mathcal{X}' \cong \mathcal{X}'' \times \mathbb{R}^{k-1}$ for some (static) metric cone $\mathcal{X}''$.

The P^*-parabolic dimension of $\mathcal{S}^k$ is $\leq k$.

Quantitative Strata of Ricci Flows

Let $(M^n, (g_t)_{t\in[-T,0]}$ be a Ricci flow with $T \geq \epsilon^{-1}$.

Definition (Bamler 2020)

A point $(x_0, 0)$ is strongly $(k, \epsilon, 1)$-symmetric if there exists $y : M \times [-\epsilon^{-1}, -\epsilon] \to \mathbb{R}^k$ such that:

- $(\partial_t - \Delta)y_j = 0$,
- $\int_{-\epsilon^{-1}}^{-\epsilon} \int_M |\langle \nabla y_i, \nabla y_j \rangle - \delta_{ij}| \, d\nu_{x_0,0;t} dt \leq \epsilon$.

$(x_0, 0)$ is $(\epsilon, 1)$-selfsimilar if, writing $d\nu_{x_0,0;t} = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg_t$, we have:

- $\int_{-\epsilon^{-1}}^{-\epsilon} \tau \int_M \left| Rc + \nabla^2 f - \frac{1}{2\tau} g \right|^2 d\nu_{x_0,0;t} dt \leq \epsilon$,
- $\sup_{t\in[-\epsilon^{-1},-\epsilon]} \int_M \left| \tau(R + 2\Delta f - |\nabla f|^2) + f - n - W \right| d\nu_{x_0,0;t} \leq \epsilon$,
- $\inf_{M\times[-\epsilon^{-1},-\epsilon]} R \geq -\epsilon$.

y_i are approximate component functions of a splitting $M' \times \mathbb{R}^k$. f is approximately the potential function of a shrinking gradient Ricci soliton

Quantitative Strata of Ricci Flows

Let $\mathcal{X}$ be a metric flow which is an $\mathbb{F}$-limit of noncollapsed smooth flows $(M_i, (g_{i,t})_{t\in[-T_i,0]})$.

Definition

$x \in \mathcal{X}$ is weakly (k, ϵ, r)-symmetric if there are $(x_i, t_i) \in M_i \times [-T_i, 0]$ converging to x which are $(\epsilon, 1)$-selfsimilar, and either:
(i) (k, ϵ, r)-split,
(ii) $(k-2, \epsilon, r)$-split and (ϵ, r)-static.
The quantitative stratum $\mathcal{S}^{\epsilon,k}_{r_1,r_2}$ is the set of $x \in \mathcal{X}$ which are not (k, ϵ, r)-symmetric for any $r \in [r_1, r_2]$.

- $\mathcal{S}^{\epsilon,k}_{r_1,r_2}$ are generally nonempty even for smooth Riemannian manifolds.
- Bamler used $\mathcal{S}^{\epsilon,k}_{r_1,r_2}$ and ϵ-regularity to get L^p estimates for Rm.
- $\mathcal{S}^k = \bigcup_{\epsilon>0} \bigcap_{r\in(0,\epsilon)} \mathcal{S}^{\epsilon,k}_{r,\epsilon}$.

Improved Splitting of Kähler Metric Flows

Let $\mathcal{X}$ be a metric flow which is an $\mathbb{F}$-limit of smooth Kähler-Ricci flows $(M_i, (g_{i,t})_{t\in[-T_i,0]})$ with $\mathcal{N}_{x,t}(1) \geq -Y$.

Theorem (H.-Jian 2022)

(*i*) $\mathcal{S}^{2k+1} = \mathcal{S}^{2k}$.
(*ii*) $\mathcal{S}^{\epsilon,2k+1}_{r_1,r_2} \subseteq \mathcal{S}^{\delta,2k}_{r_1,r_2}$, *where* $\delta = \delta(Y, \epsilon)$.

What needs to be shown: If a singular soliton (almost) splits $\mathbb{R}^{2k+1}$, then it (almost) splits $\mathbb{R}^{2k+2}$.

Corollary (H.-Jian 2022)

The P^*-parabolic dimension of $\mathcal{S}^{2k+1}$ is at most $2k$. If $\mathcal{X}$ is a metric soliton, then the usual (via Gromov-Hausdorff tangent cones) stratum $\mathcal{S}^{2k+1}$ has Minkowski dimension at most $2k$.

Idea of the Proof of Improved Splitting

Consider a gradient Kähler-Ricci soliton (M, g, J, f) which splits $\mathbb{R}$ via $y : M \to \mathbb{R}$. Then

$$z := 2\langle \nabla f, J\nabla y \rangle$$

satisfies

$$\nabla z = 2\langle \nabla^2 f, J\nabla y \rangle = J\nabla y - 2Rc(J\nabla y) = J\nabla y.$$

If ∇z is complete, the flows of $\nabla y, \nabla z$ restricted to $(y, z)^{-1}(0, 0)$ give a Riemannian splitting $M \cong M' \times \mathbb{R}^2$.
Main technical difficulty: ∇z need not (a priori) be complete.
Our approach: Construct z on closed Ricci flows, where y is an almost-splitting map.
New technical difficulty: ∇z is not sufficiently regular for almost-splitting maps (f does not satisfy a heat-type equation).

Strong Almost-Soliton Potentials

Let $(M^n, (g_t)_{t\in[-T,0]})$ be a Ricci flow with $T \geq \epsilon^{-1}$.

Definition (H.-Jian 2022)

A strong $(\epsilon, 1)$-soliton potential based at $(x_0, 0)$ is a function $h \in C^\infty(M \times [-\epsilon^{-1}, -\epsilon])$ such that if $W := \mathcal{N}_{x_0,t_0}(1)$, then:

- $(\partial_t - \Delta)(4\tau(h - W)) = -2n$,
- $\int_{-\epsilon^{-1}}^{-\epsilon} \tau \int_M \left| Rc + \nabla^2 h - \frac{1}{2\tau} g \right|^2 d\nu_{x_0,0;t} dt \leq \epsilon$,
- $\sup_{t\in[-\epsilon^{-1},-\epsilon]} \int_M \left| \tau(R + 2\Delta h - |\nabla h|^2) + h - n - W \right| d\nu_{x_0,0;t} \leq \epsilon$,
- $\int_{-\epsilon^{-1}}^{-\epsilon} \int_M \left| \tau(R + |\nabla h|^2) + (h - n) \right| d\nu_{x_0,0;t} dt \leq \epsilon$.

Most of the above are satisfied by f in almost-selfsimilar regions, where $d\nu_{x_0,0;t} = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg_t$.

Existence of Strong Almost-Soliton Potentials

Proposition (H.-Jian 2022)

If $\delta \leq \overline{\delta}(\epsilon, Y)$, then any $(\delta, 1)$-selfsimilar point admits a strong $(\epsilon, 1)$-potential function h with

$$\sup_{t \in [-\epsilon^{-1}, -\epsilon]} \int_M |f-h|^2 d\nu_t + \int_{-\epsilon^{-1}}^{-\epsilon} \int_M |\nabla (f-h)|^2 d\nu_t dt \leq \epsilon.$$

Rough idea of the proof: $4\tau(h-W)$ is defined by solving

$$\begin{cases} (\partial_t - \Delta)(4\tau(h-W)) & = -2n \\ h(\cdot, t^*) & = f(\cdot, t^*) \end{cases}.$$

Estimates for f imply h remains close to f.
Key Identity: (Heat eq version of Perleman's differential Harnack)

$$(\partial_t - \Delta)\left(\tau\left(\tau(R + 2\Delta h - |\nabla h|^2) + h - n\right)\right) = 2\tau^2 \left| Rc + \nabla^2 h - \frac{1}{2\tau} g \right|^2.$$

Constructing new Almost-Splitting Functions

Proposition (H.-Jian 2022)

If $y : M \times [-\delta^{-1}, -\delta] \to \mathbb{R}$ is a $(1, \delta, 1)$-splitting map, h is strong $(\delta, 1)$-soliton potential, and

$$z := \frac{1}{2}\langle \nabla 4\tau(h - W), J\nabla y\rangle,$$

then (y, z) is a $(2, \epsilon, 1)$-splitting map.

Aspects of the proof: z almost solves the heat equation:

$$\Box z = 2\tau\langle \nabla^2 h, \nabla(J\nabla y)\rangle \approx 0$$

$$\langle \nabla y, \nabla z\rangle \approx 2\tau\left(\frac{1}{2}g - Rc\right)(J\nabla y, \nabla y) = 0.$$

The hard part is showing that $|\nabla z|^2 \approx 1$ in the L^1 sense, since it is not clear that $Rc(\nabla y, \nabla y) \approx 0$ in the L^2 sense.

Natural Isometric Action on Singular Kähler-Ricci Solitons

If (M, g, J, f) is a Kähler-Ricci soliton, and $\omega := g(J\cdot, \cdot)$, then $\mathcal{L}_{J\nabla f}\omega = 0$ and $\mathcal{L}_{J\nabla f}J = 0$, so $\mathcal{L}_{J\nabla f}g = 0$. If $J\nabla f$ is complete, get an isometric torus action.

Theorem (H.-Jian 2022)

If $\mathcal{X}$ is a metric flow which is an $\mathbb{F}$-limit of Kähler-Ricci flows, then any time slice (X, d) admits an isometric action whose restriction to the regular set is given by the flow of $J\nabla f$.

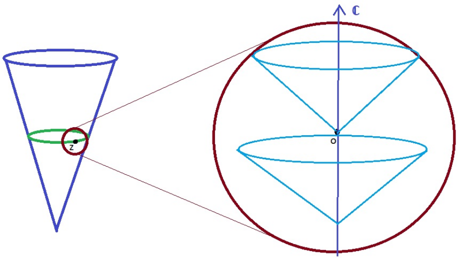
Recurring technical difficulty: $J\nabla f$ need not be complete.
Recurring solution: Approximate by smooth, closed Kähler-Ricci flows (f will be approximated by strong almost-soliton potentials). Show the heat kernel satisfies approximate infinitesimal symmetry, and pass to the limit.

The Static Case

Theorem (H.-Jian 2022)

If in addition $\mathcal{X}$ is static (hence a metric cone), then the action is locally free away from the vertex.

Proof sketch: If $X = C(\Sigma)$ and $z \in \Sigma$ is fixed by the action, then the induced action on a tangent cone based at z preserves distance to the vertex, but is also induced by a splitting.



Thank you for your attention.