1 Backwards Uniqueness for the Heat Equation in an Exterior Domain

(after L. Escauriaza, G. Seregin, and V. Sverak [?])

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Abstract

Two L^2 Carleman estimates are proved for the backwards heat operator in $(\mathbb{R}^n \setminus B_R) \times [0,T]$. These estimates are then used to prove backwards uniqueness for functions satisfying general growth conditions and solving a heat-type equation.

1.1 Introduction

The focus of the paper is on solutions of the backwards heat equation on the exterior domain $Q_{R,T} := (\mathbb{R}^n \setminus \bar{B}_R) \times [0,T]$. The main goal is the following theorem.

Theorem 1. *Suppose $u \in C^\infty(Q_{R,T})$ satisfies $u(\cdot,0) = 0$ in $\mathbb{R}^n \setminus \bar{B}_R$ and*

$$|(\partial_t + \Delta)u| \le C(|u| + |\nabla u|), \qquad |u(x,t)| \le Ce^{C|x|^2}$$

in $Q_{R,T}$. Then $u = 0$ in $Q_{R,T}$.

Remark 2. *A similar problem was previous solved independently by C.C. Poon and X.Y. Chen, where $Q_{R,T}$ is replaced by $\mathbb{R}^n \times [0,T]$, without using Carleman estimates. That result is a corollary of the main theorem here by letting $R \to 0$.*

Remark 3. *Note that this theorem is false, even with stronger assumptions, if $\partial_t + \Delta$ is replaced by $\partial_t - \Delta$, as the heat kernel $p(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}$ is a counterexample.*

1.2 Carleman-Type Inequalities for the Backwards Heat Equation

The first goal is to prove a L^2 Carleman inequality for the backwards heat operator. In particular, we want to bound weighted L^2 norms of $u \in C_c^\infty(\mathbb{R}^n \times [0,1))$ and $|\nabla u|$ in terms of the weighted L^2 norm of $(\partial_t + \Delta)u$.

Theorem 4. *(First Carleman Estimate)There exists* $\alpha_0 = \alpha_0(R, n) < \infty$ *such that for all* $\alpha \geq \alpha_0$ *and all* $u \in C_c^\infty(Q_{R,T})$ *satisfying* $u(\cdot, 0) = 0$*, we have*

$$||e^{\alpha(T-t)(|x|-R)+|x|^2}u||_{L^2(Q_{R,T})} + ||e^{\alpha(T-t)(|x|-R)+|x|^2}\nabla u||_{L^2(Q_{R,T})}$$
$$\leq ||e^{\alpha(T-t)(|x|-R)+|x|^2}(\partial_t u + \Delta u)||_{L^2(Q_{R,T})} + ||e^{|x|^2}\nabla u(\cdot, T)||_{L^2(\mathbb{R}^n \setminus \bar{B}_R)}.$$

For $G \in C^\infty(Q_{R,T})$, define $F := (\partial_t G - \Delta G)/G$. The $L^2(Gdxdt)$-self-adjoint part of $\partial_t + \Delta$ is

$$S = \Delta + \nabla \log G \cdot \nabla - \frac{1}{2}F,$$

and the $L^2(Gdxdt)$-skew-adjoint part of $\partial_t + \Delta$ is

$$A = \partial_t - \nabla \log G \cdot \nabla + \frac{1}{2}F.$$

We may compute the principal symbol of the commutator: by [**?**],

$$\sigma_{[S,A]}(x, t, \xi, \tau) = \{\sigma_S, \sigma_A\}(x, t, \xi, \tau) = -\nabla^2 \log G(\xi, \xi),$$

which suggests that (up to lower order terms) if G is log-convex, it should be possible to prove a priori estimates for $[S, A]$. In fact, for carefully chosen G, an elementary integration by parts argument shows that $u \mapsto \langle Su, Au\rangle_{L^2(Gdxdt)}$ has strong positivity properties:

$$\langle Su, Au\rangle_{L^2(Gdx)}(t) = \frac{1}{2}\int_{\mathbb{R}^n \setminus B_R} u^2(\partial_t F + \Delta F)Gdx - \int_{\mathbb{R}^n \setminus \bar{B}_R} |\nabla u|^2 Gdx \quad (1)$$
$$+ 2\int_{\mathbb{R}^n \setminus B_R} \nabla^2 \log G(\nabla u, \nabla u)Gdx - \int_{\mathbb{R}^n \setminus \bar{B}_R} u^2 FGdx.$$

Since $||(\partial_t+\Delta)u||^2_{L^2(Gdxdt)} = ||Au||^2_{L^2(Gdxdt)}+||Su||^2_{L^2(Gdxdt)}+2\langle Au, Su\rangle_{L^2(Gdxdt)}$, the desired Carleman inequality will follow (by integrating (**??**) from $t = 0$ to $t = T$) from finding G such that $F \leq 0$, $\partial_t F + \Delta F \geq 1$, and $\nabla^2 \log G \geq I$. The functions $G(x, t) := e^{2\alpha(T-t)(|x|-R)+2|x|^2}$ satisfy these properties for large $\alpha > 0$. The parameter α will be important later for dealing with our lack of information about u near $(\partial B_R) \times [0, T]$, making use of the important fact that, for fixed $t_1 < t_2$, $G(x, t_1)/G(x, t_2) \to \infty$ as $\alpha \to \infty$.

For the second Carleman estimate, we need to define the auxiliary functions $\sigma(t) := te^{-\frac{t}{3}}$ and $\sigma_a(t) := \sigma(t + a)$.

Theorem 5. *(Second Carleman Estimate) There exists $N = N(n) < \infty$ such that, for any $\alpha \geq 0$, $a \in (0,1)$, $y \in \mathbb{R}^n$, and $u \in C_c^\infty(\mathbb{R}^n \times [0,1))$ satisfying $u(\cdot, 0) \equiv 0$, we have*

$$||\sigma_a^{\alpha-1/2} e^{-\frac{|x-y|^2}{8(t+a)}} u||_{L^2(\mathbb{R}^n\times(0,1)} + ||\sigma_a^{\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} \nabla u||_{L^2(\mathbb{R}^n\times(0,1)}$$
$$\leq N||\sigma_a^{\alpha} e^{-\frac{|x-y|^2}{8(t+a)}} (\partial_t u + \Delta u)||_{L^2(\mathbb{R}^n\times(0,1)}.$$

Note that in this case, the Gaussian weight $x \mapsto e^{-\frac{|x-y|^2}{8(t+a)}}$ is in fact log-concave rather than log-convex, so the σ^α term is essential for the second Carleman estimate. Set $G_a(x,t) := (4\pi(t+a))^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t+a)}}$, and apply (**??**) with G replaced by $\sigma_a^{-\alpha} G_a$. Assuming that $u(\cdot,0) \equiv 0$ and $u \in C_c^\infty(\mathbb{R}^n \times [0,1)$, multiplying (**??**) by $\sigma/\dot{\sigma}$, and integrating from $t = 0$ to $t = 1$ leads to

$$\int_{\mathbb{R}^n\times[0,1]} \frac{\sigma_a^{1-\alpha}}{\dot{\sigma}_a}(Su)(Au)G_a dxdt \tag{2}$$
$$= \int_{\mathbb{R}^n\times(0,1)} \frac{\sigma_a^{1-\alpha}}{\dot{\sigma}_a} \left(\left(\log(\frac{\sigma_a}{\dot{\sigma}_a}) \right)' I + 2\nabla^2 \log G_a \right) (\nabla u, \nabla u) G_a dxdt.$$

We used here that G_a is an exact solution of the heat equation. Because (this is where the e^{-3t} term in σ is important)

$$\left(\log(\frac{\sigma_a}{\dot{\sigma}_a}) \right)' I + 2\nabla^2 \log G_a = \frac{1}{3} \frac{1/3}{1-(t+a)/3)} I \geq \frac{1}{3} I$$

and $\frac{1}{3e} \leq \dot{\sigma}_a(t) \leq 1$ for $t \in [0,1]$, we conclude that

$$||\sigma_a^{-\alpha} G_a^{\frac{1}{2}} \nabla u||_{L^2(\mathbb{R}^n\times[0,1])} \leq N'||\sigma_a^{-\alpha} G_a^{\frac{1}{2}} (\partial_t u + \Delta u)||_{L^2(\mathbb{R}^n\times[0,1])}.$$

where we replaced α with $1 + 2\alpha$. Integrating $(\partial_t + \Delta)u^2 = 2u(\partial_t + \Delta u) + 2|\nabla u|^2$ against the weight $\sigma_a^{-2\alpha} G_a$, and integrating by parts and using Cauchy-Schwartz gives the claim. Note that here the nice formula for the heat operator applied to the weights rescues us, when a Poincare inequality in the unbounded domain would not be helpful.

1.3 Proving Quadratic Exponential Decay

We first recall two a priori estimates for solutions of second order parabolic equations.

Lemma 6. *(Gradient Estimate for the Nonhomogeneous Heat Equation) Set $P_r := B_r \times [-r^2, 0)$ for $r > 0$. There exists $C^* = C^*(n) < \infty$ such that, for any $u \in C^\infty(P_r)$, we have*

$$\sup_{(x,t)\in P_r} d_{x,t}|\nabla u(x,t)| \le C^* \sup_{(x,t)\in P_r} (|u(x,t)| + d_{x,t}^2 |(\partial_t + \Delta)u(x,t)|),$$

where $d_{x,t} := \min\{d(x, \partial B_r), |t|\}$.

Proof. Use a parabolic barrier function and the maximum principle. □

Lemma 7. *(Parabolic Mean Value Inequality) There exists $C = C(n) < \infty$ such that, for any $s > 0$ and $u \in C^\infty(B_{\sqrt{s}}(y) \times [s, 2s])$ satisfying $|(\partial_t + \Delta)u| \le |u| + |\nabla u|$, we have*

$$|u(y,s)|^2 \le \frac{C}{s^{n+2}} \int_s^{2s} \int_{B_{\sqrt{s}}(y)} u^2 dx dt.$$

Proof. Use parabolic Moser iteration on u_+, u_-. □

Lemma 8. *There exists $\epsilon = \epsilon(n) > 0$ and $M = M(n) < \infty$ such that the following holds. Suppose $u \in C^\infty(Q_{R,1})$ satisfies*

$$|\partial_t u + \Delta u| \le \epsilon(|u| + |\nabla u|), \qquad |u(x,t)| \le e^{\epsilon |x|^2},$$

and $u(\cdot, 0) = 0$ in $\mathbb{R}^n \setminus \bar{B}_R$ for some $R \ge 1$. Then

$$|u(y,s)| + |\nabla u(y,s)| \le M e^{-\frac{|y|^2}{Ms}} (1 + ||u||_{L^\infty((B_{4R} \setminus B_R)\times(0,1))}) \qquad \text{for } (y,s) \in Q_{6R, M^{-1}}.$$

Proof. Fix $y \in \mathbb{R}^n \setminus 6R$. The strategy is to first obtain an $L^2(dxdt)$ estimate on a forwards parabolic cylinder around y, by applying the second Carleman inequality with the Gaussian weight centered at y. This gives a good L^2 bound for u near y since the Gaussian weight is bounded below near y, while $\sigma^{-\alpha}(t)$ is large for t small. The claim then follows immediately from the above parabolic regularity theorems.

To get a right hand side of the Carleman inequality we can estimate effectively, we need to cutoff u appropriately, so that we only have to estimate $G_a\sigma_a^{-2\alpha}u^2$ where the weight $\sigma_a^{-2\alpha}G_a$ is relatively small. Define $u_r(x,t) = u(x,t)\varphi(t)\psi_r(x)$, where $\phi \in C^\infty(\mathbb{R})$ and $\psi_r \in C_c^\infty(\mathbb{R}^n)$ satisfy $\phi = 1$ on $(-\infty, 1/2]$, $\phi = 0$ on $[3/4,\infty)$, $\psi_r = 1$ on $B_{2r}\setminus B_{3R}$, $\psi_r = 0$ outside $B_{3r}\setminus B_{2R}$. Then

$$|(\partial_t+\Delta)u_r| \leq \epsilon(|u_r|+|\nabla u_r|)+|\varphi' u|+\varphi(|u|(|\Delta\psi_r|+|\nabla\psi_r|)+2|\nabla\psi_r|\cdot|\nabla u|),$$

so applying the first Carleman estimate gives

$$\begin{aligned}||\sigma_a^{-\alpha-\frac{1}{2}}G_a u_r||_{L^2(\mathbb{R}^n\times[0,1])} + ||\sigma_a^{-\alpha}G_a\nabla u_r||_{L^2(\mathbb{R}^n\times[0,1])}\\ \leq C(n)(||\sigma_a^{-\alpha}G_a u||_{L^2((\mathbb{R}^n\setminus B_R)\times[\frac{1}{2},\frac{3}{4}]} + ||\sigma_a^{-\alpha}G_a(|u|+|\nabla u|)||_{L^2((A_1\cup A_2)\times[0,3/4]}),\end{aligned}$$

where $A_1 := B_{3R}\setminus B_{2R}$ and $A_2 := B_{3r}\setminus B_{2r}$. Now apply the gradient estimate for u to get (at scale 1, and assuming $\epsilon < \frac{1}{2}C^*$) to get $|\nabla u(x,t)| \leq C(n)e^{\epsilon|x|^2}$, so the integral over A_2 vanishes as we let $r\to\infty$, for ϵ small. Also, we know $y \notin A_1$, so the right hand side stays bounded as $a\to 0$, and the left hand side converges by the monotone convergence theorem. Applying the gradient estimate in A_1 gives $M(n) < \infty$ such that

$$||\sigma^{-\alpha}G^{\frac{1}{2}}(|u|+|\nabla u|)||_{L^2(A_1\times[0,3/4])} \leq M^\alpha\left(\sup_{t>0} t^{-\alpha}e^{-\frac{|y|^2}{16t}}\right)||u||_{L^\infty((B_{4R}\setminus B_R)\times[0,1]}),$$

and completing the square gives

$$||\sigma^{-\alpha}G^{\frac{1}{2}}u||_{L^2((\mathbb{R}^n\setminus B_R)\times[\frac{1}{2},\frac{3}{4}]} \leq M^\alpha e^{|y|^2}.$$

By Stirling's formula,

$$\sup_{t>0} t^{-k}e^{-\frac{|y|^2}{16t}} = |y|^{-2k}(16k)^k e^{-k} \leq |y|^{-2k}M^k k!,$$

so we can take $\alpha = k$, multiply by $|y|^{2k}(2M)^{-k}/k!$, and sum to get

$$||e^{\frac{|y|^2}{4Mt}}G^{\frac{1}{2}}u||_{L^2(Q_{3R,(8M)^{-1}})} \leq C(n)(1+||u||_{L^\infty((B_{4R}\setminus B_R)\times[0,1]}).$$

□

1.4 Completing the Proof of Theorem 1

Lemma 9. *With the same hypotheses as the previous lemma, we have* $u = 0$ *in* $Q_{R,\epsilon}$.

Proof. We now apply the second Carleman inequality to $u_{a,r} = u\psi_{a,r}$, where $\psi_{a,r} \in C_c^\infty(B_{2r} \setminus B_{(1+a)R})$, $|\nabla\psi a, r| \leq C(n)a^{-1}$, and $\psi_{a,r} = 1$ on $B_r \setminus B_{(1+2a)R}$. Take $T = 4\epsilon$ to get

$$\begin{aligned} e^{10\alpha\epsilon aR}||u||_{L^2((B_r \setminus B_{(1+10a)R}) \times [0,\epsilon]} &\leq C(n)e^{8\alpha\epsilon r+4r^2}|||u| + |\nabla u|||_{L^2((B_{2r} \setminus B_r) \times [0,4\epsilon])} \\ &+ C(a,n)e^{8\alpha\epsilon aR}|||u| + |\nabla u|||_{L^2((B_{(1+2a)R} \setminus B_{(1+a)R}) \times [0,4\epsilon]} \\ &+ C(n)||e^{|x|^2}(|u| + |\nabla u|)||_{L^2(B_{\mathbb{R}^n \setminus B_R})}. \end{aligned}$$

Note that we traded integrating over a larger region in return for a better exponent on the left hand side. Dealing with the gradient terms as before, we obtain

$$||u||_{L^2((B_r \setminus B_{(1+10a)R}) \times [0,\epsilon])} \leq C \cdot (e^{\alpha r - r^2} + e^{-2\alpha\epsilon aR}).$$

Let $r \to \infty$, then $\alpha \to \infty$, then $a \to 0$. □

Proof of Theorem 1 Now we finish the proof of Theorem 1. By parabolic rescaling, we can assume the hypotheses of the previous lemmas (including $T \geq 1$), so $u = 0$ on $Q_{R,\epsilon}$. Repeat with ϵ as the new initial time, and keep repeating to get $u = 0$ on $Q_{R,a}$, where $T - a < 1$. Rescaling so that a is the initial time and T becomes 1, and applying the previous lemma gives $u = 0$ on $Q_{R,a+(T-a)\epsilon}$. Iterate, and see that we have $u = 0$ outside some region whose time interval is decreasing geometrically, hence $u = 0$ on all of $Q_{R,T}$. □

References

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