Ricci Flow with Ricci Curvature and Volume Bounded Below

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Outline

Ricci Flow

Definition

A smooth, 1-parameter family of Riemannian metrics $(g_t)_{t\in[0,T)}$ on a closed manifold M^n satisfies Ricci flow if

$$\partial_t g_t = -2Rc(g_t),$$

where $Rc(g_t)$ is the Ricci curvature.

Ricci flow was first used by Richard Hamilton in 1982, to prove that every 3-dim Riemannian manifold with positive Ricci curvature is a space form.

Basic Facts About Ricci Flow

- (PDE Classification) Ricci flow is a second-order nonlinear weakly parabolic system. In harmonic coordinates,

$$\partial_t g_{t,ij} = -2Rc(g_t)_{ij} = \Delta g_{t,ij} + Q_{ij}(g_t, Dg_t).$$

- (Short-time existence/uniqueness) For any smooth Riemannian metric g on M, there is a unique solution $(M^n, (g_t)_{t\in[0,T)})$ of Ricci flow with $g_0 = g$.
- (Evolution of Curvature) The curvature tensor under Ricci flow satisfies a (strictly) parabolic PDE with nonlinear reaction term:

$$\partial_t Rm = \Delta Rm + Q(Rm).$$

Ricci Solitons

- Up to scaling $\sigma(t)$ and diffeomorphisms (φ_t), "fixed points" of Ricci flow are self-similar solutions:

$$g(t) = \sigma(t)\varphi_t^* g.$$

- Such solutions are equivalent to a fixed metric g and a vector field X satisfying the Ricci soliton equation:

$$Rc(g) + \mathcal{L}_X g = \lambda g.$$

- If $X = \nabla f$, then (M, g, f) is a gradient Ricci soliton.
- Different properties when $\lambda < 0, \lambda = 0, \lambda > 0$ (expanding, steady, shrinking).
- Examples: Einstein manifolds, shrinking cylinder, 2D asymptotically conical expanders (Gutperle-Headrick-Minwalla-Schomerus 2002)

How Singularities Form

Suppose $(M^n, (g_t)_{t\in[0,T)})$ is a closed Ricci flow which cannot be extended past some time $T \in (0, \infty)$.

Theorem (Hamilton, 1982)

$$\lim_{t\to T} \max_M |Rm|(\cdot, T)(T-t) > 0.$$

Theorem (Šešum, 2003)

$$\limsup_{t\to T} \max_M |Rc|(\cdot, t) = \infty.$$

Theorem (Wang, 2007)

If $Rc(g_t) \geq -Ag_t$ for all $t \in [0, T)$, then

$$\int_0^T \int_M |R|^{\frac{n+2}{2}} dg_t dt = \infty.$$

Singularity Models

Classical Approach: Find points $(p_i, t_i) \in M \times [0, T)$ where $Q_i := |Rm|(p_i, t_i) \to \infty$, but such that the rescaled solutions $\widetilde{g}_t^i := Q_i g_{t_i + Q_i^{-1} t}$ have bounded curvature. Use Cheeger-Gromov-Hamilton compactness to extract a singularity model

$$(M_\infty, g_t^\infty, p_\infty) = \lim_{t \to \infty} (M, \widetilde{g}_t^i, p_i),$$

which often has nice properties.

Examples: If $n = 3$, κ-solutions appear. For Type-I solutions, shrinking GRS. If $|R| \leq C$ and $n = 4$, Ricci-flat ALE space.

Shortcoming: When $n \geq 4$, difficult to extract information at different scales.

Modern Approach: Develop compactness theory with less stringent curvature assumptions.

Ricci Flow with Ricci Curvature and Volume Bounded Below

Suppose $(M^n, (g_t)_{t\in[0,T)})$ is a closed Ricci flow with $Rc(g_t) \geq -Ag_t$ and $|M|_{g_t} > A^{-1}$ for all $t \in [0, T)$.
Question: Can a finite-time singularity develop at time $t = T$?

Theorem (Zhang, 2012)

No singularity if g_0 is Kahler.

Some easy cases:

- No singularity if $n \leq 3$
- No singularity if Type-I
- No singularity if rotationally symmetric

Noncollapsed Ricci Limit Spaces

Theorem (Anderson, Cheeger, Colding, Gromov, Naber, Tian)

If (M_i^n, g_i, p_i) is a sequence of pointed Riemannian manifolds with $Rc(g_i) = \lambda_i g_i$, $|\lambda_i| \leq 1$, and $|B(p_i, 1)| > \nu > 0$, then a subsequence converges in the pointed Gromov-Hausdorff sense to a complete metric length space, which has the structure of a C^∞ Riemannian manifold away from a subset of codimension 4. Convergence is smooth on the regular set.

- If $|Rc(g_i)| \leq 1$ instead of Einstein, the regular set is only $C^{1,\alpha}$
- If only $Rc(g_i) \geq -g_i$, then the regular set is more complicated, not always open, and singularities have codimension 2

Theorem (Chen, Yuan, 2017)

If (M_i, g_i) are time-slices of noncollapsed Ricci flows with Ricci curvature bounded below, then the regular set is open and smooth.

A Nonsmooth Limit at the Singular Time

$(M^n, (g_t)_{t\in[0,T)})$ a closed Ricci flow with $Rc(g_t) \geq -Ag_t$, $|M|_{g_t} \geq A^{-1}$. Then

$$(X,d) = \lim_{t\to T}(M, d_{g_t})$$

exists in the Gromov-Hausdorff sense, and $X = M/\sim$, where

$$x \sim y \iff d_{g_T}(x,y) := \lim_{t\to T} d_{g_t}(x,y) = 0.$$

(X,d) is a noncollapsed Ricci-limit space with smooth, open regular part. Its regular part corresponds to the points $M \setminus \Sigma$ where the curvature remains bounded.

Basic Heuristic (Chen-Yuan): (M, d_{g_t}) should degenerate like a sequence of Einstein manifolds.

Refined Heuristic: (M, d_{g_t}) should degenerate like a sequence of Einstein manifolds at the Type-I scale and below.

Question: Does X have singularities of codimension 4?

$\mathbb{F}$-Convergence

Definition (Wasserstein Distance)

For measures μ, ν on a metric space (X, d), the Wasserstein distance is

$$d_{W_1}(\mu, \nu) := \inf_{\pi} \int_{X \times X} d(x, y) d\pi(x, y),$$

where the infimum is taken over all couplings π of (μ, ν).

Gromov-Wasserstein distance: When measures do not live on the same metric space, take infimum over the Wasserstein distance over all isometric embeddings into a common metric space.

$\mathbb{F}$-**Convergence:** A sequence $(M_i, (g_t)_{t \in I^i}, (\nu_t^i)_{t \in I^i})$ of Ricci flows with reference conjugate heat flows $\mathbb{F}$-**converges** if the time slices converge in the Gromov-Wasserstein distance at almost-every time.

Tangent Flows

Notation: If $K(x,t;\cdot,\cdot)$ is the conjugate heat kernel based at (x,t), we write $d\nu_{x,t;s} = K(x,t\cdot,s)dg_s$.
Let $(M^n,(g_t)_{t\in[0,T)})$ be a closed Ricci flow with the conjugate heat flow $\nu_{x,T;t} := \lim_{t_i\nearrow T}\nu_{x,t_i;t}$ based at the singular time.

Theorem (Bamler, 2020)

If $\tau_i \searrow 0$, $g_t^i := \tau_i^{-1}g_{T+\tau_i t}$, $\nu_t^i := \nu_{x,T;T+\tau_i t}$, then

$$(M,(g_t^i)_{t\in[-\tau_i T,0)},(\nu_t^i)_{t\in[-\tau_i T,0)}) \to (\mathcal{X},(\mu_t)_{t<0}),$$

in $\mathbb{F}$-distance, where $\mathcal{X}$ is a metric flow corresponding to a singular soliton $(X,d,\mathcal{R},g,f)$ with singularities of codimension 4, and

$$d\mu_t = (4\pi\tau)^{-\frac{n}{2}}e^{-f}dg_t.$$

A Criterion for Gromov-Hausdorff Convergence

Proposition

Let (X_i, d_i, μ_i) be a sequence of metric measure spaces converging in the Gromov-Wasserstein sense to $(X_\infty, d_\infty, \mu_\infty)$, and suppose $x_i \in X_i$ are such that, for any $D < \infty$, $r > 0$, there exists $\epsilon = \epsilon(D, r) > 0$ such that for all $y_i \in B(x_i, D)$, we have

$$\mu_i(B(x_i, r)) \geq \epsilon.$$

Then there exists $x_\infty \in X_\infty$ such that $(X_i, d_i, x_i) \to (X_\infty, d_\infty, x_\infty)$ in the pointed Gromov-Hausdorff sense.

We will take the x_i to be H_n-centers.

Theorem (Bamler, 2020)

For any conjugate heat kernel $(\nu_{x,t;s})_{s\in[-2,t)}$ on a closed Ricci flow $(M, (g_t)_{t\in[-2,0]})$ and $s \in [-2, t)$, there exists $z \in M$ such that $\int_M d^2(z, y) d\nu_{x,t;s}(y) \leq H_n(t - s)$. (z, s) is called an H_n-center of (x, t).

A Weak Heat Kernel Lower Bound

Let $(M,(g_t)_{t\in[-4,0]})$ be a closed Ricci flow, such that $Rc(g_t)\geq -Ag_t$ and $|B(x,t,r)|_{g_t}\geq A^{-1}r^n$ for all $r\in(0,1]$, $(x,t)\in M\times[-4,0]$.

Proposition (H, 2021)

For any $D<\infty$ and $\delta>0$, there exists $\sigma=\sigma(A,D,\delta)>0$ such that the following holds. Given any H_n-center $(y,-2)\in M\times\{-2\}$ of $(x,0)$, there is a subset $S\subseteq B(y,-2,D)$ such that

$$\inf_S K(x,0;\cdot,-2)>\sigma,$$

$$|B(y,-2,D)\setminus S|_{g_{-2}}<\delta.$$

Cheeger-Jiang-Naber Estimates

Definition: $r_{Rm}(x,t) := \sup\{r > 0; |Rm|(\cdot,t) < r^{-2} \text{ on } B(x,t,r)\}$.

Theorem (Codimension 1 ϵ-Regularity)

There exists $\epsilon_0 = \epsilon_0(A) > 0$ such that if $(M^n, (g_t)_{t\in[-2,0]})$ is a closed Ricci flow with $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$, for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$, and if

$$d_{GH}\left(B(x,0,\epsilon_0^{-1}r), B\left((0^{n-1},z_*),\epsilon_0^{-1}r\right)\right) < \epsilon_0 r$$

for some metric cone $C(Z)$, then $r_{Rm}(x,t) \geq \epsilon_0 r$.

Corollary (Curvature Scale Estimates)

If $(M^n, (g_t)_{t\in[-2,0]})$ satisfies the same hypotheses, then

$$|\{r_{Rm}(\cdot,0) < r\} \cap B(x,0,1)|_{g_0} \leq C(n,A)r^2$$

for all $x \in M$, $r \in (0,1]$.

Proving the Weak Heat Kernel Lower Bound

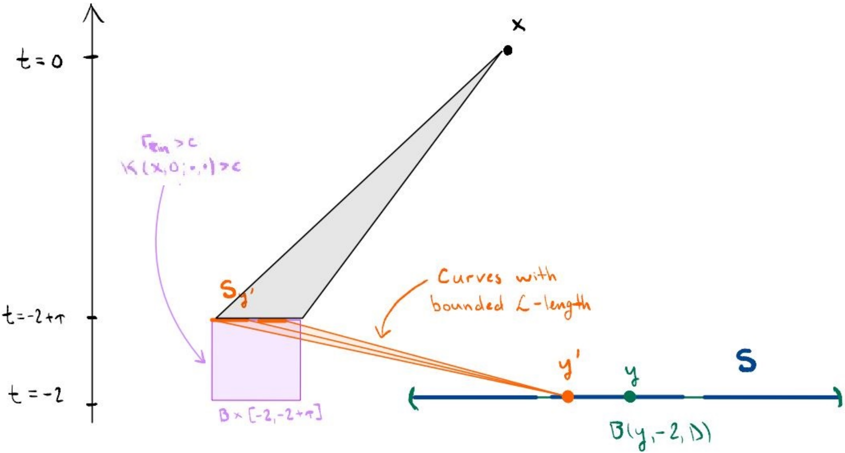
Essential Ingredients:

- Curvature scale estimates
- Bamler's variance estimate, on-diagonal heat kernel upper bound
- Perelman integrated differential Harnack
$K(x,t;y,s) \geq (4\pi(t-s))^{-\frac{n}{2}} e^{-\ell_{(x,t)}(y,s)}$
- Colding's segment inequality

Rough idea:
Step 1: Find a nearby subset B of definite size such that
$K(x,0;\cdot,-2+\tau) > c$, $r_{Rm}(\cdot,-2+\tau) > c$ for some small $c, \tau > 0$.

Step 2: Find curves with bounded $\mathcal{L}$-length from points in B to most points of $B(y,-2,D)$.

Proving the Weak Heat Kernel Lower Bound (Cont)



Shrinking Solitons with Nonnegative Ricci Curvature

Theorem (Ni, 2005)

If (M^n, g, f) is a non-Ricci-flat shrinking Ricci soliton with $Rc(g) \geq 0$, then $\inf_M R > 0$.

Idea of the Proof: $\langle \nabla R, \nabla f \rangle = 2Rc(\nabla f, \nabla f) \geq 0$, so it suffices to show that flowing backwards along integral curves of ∇f moves points into a fixed compact set.

Proposition (H, 2021)

If $(\mathcal{R}, g, f)$ is a singular shrinking Ricci soliton which is also a Ricci limit space, then the previous theorem applies.

This relies on $\mathcal{R}$ having **mild singularities**, and on $\nabla^2 f$ being locally bounded.

Tangent Flows are Ricci-Flat Cones

Theorem (H. 2021)

Suppose $(M^n, (g_t)_{t\in[0,T)})$ is a closed Ricci flow with $Rc(g_t) \geq -Ag_t$ and $|M|_{g_t} \geq A^{-1}$. If $x \in \Sigma$, then any tangent flow at (x, T) is a nontrivial Ricci-flat cone.

Proof:

- $\mathbb{F}$-convergence $\implies$ pointed Gromov-Hausdorff convergence
- Colding's volume convergence theorem $\implies$ singular soliton has maximal volume growth
- Adapted Ni's Theorem $\implies$ Ricci flat or scalar lower bound
- Any asymptotic cone is a noncollapsed Ricci limit space, so lower scalar bound cannot occur

Singular Points are Type-II

Proposition (H, 2021)

Suppose $(M^n, (g_t)_{t\in[0,T)})$ *is a closed Ricci flow with* $Rc(g_t) \geq -Ag_t$ *and* $|M|_{g_t} \geq A^{-1}$. *If* $x \in \Sigma$, *then*

$$\limsup_{t\to T} r_{Rm}^{-2}(x,t)(T-t) = \infty.$$

Idea of proof: Otherwise,

$$\frac{1}{\sqrt{T-t}} \int_t^T \sqrt{T-s}R(x,s)ds < \infty,$$

which implies Gaussian lower bounds for the conjugate heat kernel, so that after a Type-I rescaling, $(x,-1)$ converges to a point in $C(N) \setminus o_*$. The flow is static, so (x,t) converges to the same point, a contradiction for t close to 0.

Pointed Gromov-Hausdorff Convergence

Theorem (H, 2021)

Suppose $(M^4, (g_t)_{t\in[0,T)})$ is a closed, simply-connected Ricci flow with $Rc(g_t) \geq -Ag_t$ and $|M|_{g_t} \geq A^{-1}$. If $x \in \Sigma$, then there is a finite subgroup $\Gamma \leq O(4)$ such that

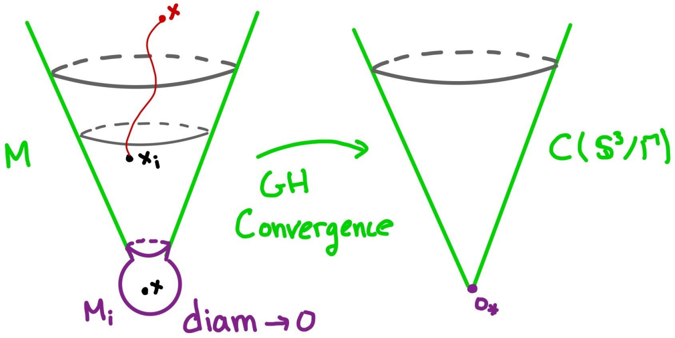
$$(M, (T-t)^{\frac{1}{2}} d_{g_t}, x) \to (C(S^3/\Gamma), o_*)$$

in the pointed Gromov-Hausdorff sense.

Proof idea: First prove for a fixed subsequence, using Gaussian heat kernel upper bound and distortion estimates via pseudolocality. Then upgrade to convergence as $t \nearrow T$. The volume lower bound implies $|\Gamma| \leq N$, but the set of $C(\mathbf{S}^3/\Gamma)$ is discrete in the Gromov-Hausdorff topology.

Proof of Pointed Gromov-Hausdorff Convergence

Integrate $K(x,0;\cdot,-1) \leq C \exp\left(-\frac{d_{g_0}^2(x,\cdot)}{C}\right)$ on $B(x_i,-1,D)$ to get $d_{g_0}(x,x_i) \leq C$. Then apply pseudolocality:



Orbifold Convergence

Theorem (H, 2021)

Suppose $(M^4, (g_t)_{t\in[0,T)})$ is a closed, simply-connected Ricci flow with $Rc(g_t) \geq -Ag_t$ and $|M|_{g_t} \geq A^{-1}$, and $(X,d) = \lim^{GH}_{t\to T}(M, d_{g_t})$. Then (X,d) is a C^0 Riemannian orbifold with finitely many conical singularities, and convergence is smooth away from these points. If $x \in \Sigma$ has tangent flow $C(S^3/\Gamma)$, then $\overline{x}$ has this as its tangent cone.

Bamler-Zhang and Simon's work implies a similar description if the lower Ricci bound is replaced by $|R| \leq A$.

Remark (Flowing through the singularity)

Simon showed that there is a well-defined orbifold Ricci flow $(\widetilde{M}, \widetilde{g}_{t\in[T,T+\epsilon)})$ with

$$\lim_{t\searrow T}(\widetilde{M}, d_{\widetilde{g}_t}) \to (X,d).$$

Codimension 2 ϵ-Regularity

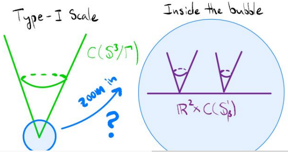
Proposition (Codimension 2 ϵ-Regularity, H, 2021)

There exists $\epsilon_0 = \epsilon_0(A, \underline{T}) > 0$ such that if $T \geq \underline{T}$, $(M^n, (g_t)_{t\in[0,T)})$ is a closed Ricci flow with $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$, for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$, and if

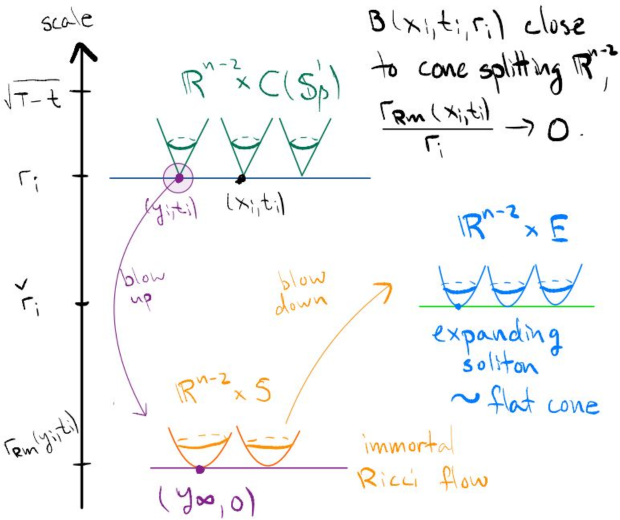
$$d_{GH}\left(B(x,t,\epsilon_0^{-1}r), B\left((0^{n-2},z_*),\epsilon_0^{-1}r\right)\right) < \epsilon_0 r$$

for some $(x,t) \in M \times [\frac{T}{2}, T)$, $r \in (0, \epsilon_0\sqrt{T-t}]$ and metric cone $C(Z)$, then $r_{Rm}(x,t) \geq \epsilon_0 r$.

The upshot (in 4 dimensions): Singularities in the bubbles have codimension ≥ 3.



Codimension 2 ϵ-Regularity Proof



Codimension 3 ϵ-Regularity

Proposition (Codimension 3 ϵ-Regularity, H, 2021)

There exists $\epsilon_0 = \epsilon_0(A, \underline{T}) > 0$ such that if $T \geq \underline{T}$, $(M^n, (g_t)_{t\in[0,T)})$ is a closed orientable Ricci flow with $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$, for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$, and if

$$d_{GH}\left(B(x,t,\epsilon_0^{-1}r), B\left((0^{n-3}, z_*), \epsilon_0^{-1}r\right)\right) < \epsilon_0 r$$

for some $(x,t) \in M \times [\frac{T}{2}, T)$, $r \in (0, \epsilon_0\sqrt{T-t}]$ and metric cone $C(Z)$, then $r_{Rm}(x,t) \geq \epsilon_0 r$.

Idea of proof: Use codimension-2 ϵ-regularity to get convergence to $\mathbb{R}^{n-3} \times C(Z)$, where Z has smooth link. A maximum principle argument shows $C(Z)$ is flat, and orientability rules out $Z = \mathbb{R}P^2$.

L^p Curvature Estimates for $p \in [0,2)$

Theorem (H, 2021)

Suppose $(M^4, (g_t)_{t\in[0,T)})$ is a closed, simply connected Ricci flow with $Rc(g_t) \geq -Ag_t$, $|M|_{g_t} \geq A^{-1}$ for $t \in [0,T)$. Then there exists $E < \infty$ such that, for any $(x,t) \in M \times [\frac{T}{2}, T)$, $r \in (0,1]$, we have

$$|\{r_{Rm}(\cdot,t) < r\}|_{g_t} \leq Er^4.$$

Corollary (H, 2021)

With the same hypothesis, we have

$$\sup_{t\in[0,T)} \int_M |Rm|^p dg_t < \infty$$

for any $p \in [0,2)$.

Obvious question: What happens when $p = 2$?

Proof of the L^p Estimates

Main Idea: Decompose each time slice $M \times \{t\}$ into three regions, and estimate each region differently.

The Inner Region: $r_{Rm}(\cdot, t) << \sqrt{T-t}$. Then combine the Cheeger-Jiang-Naber estimates with codimension 3 ϵ-regularity.

The Intermediate Region: $r_{Rm}(\cdot, t) \approx \sqrt{T-t}$. Using Type-I behavior of the flow, show these points are close to the orbifold singularities.

The Outer Region: $r_{Rm}(\cdot, t) >> \sqrt{T-t}$. Pseudolocality lets us compare the high-curvature region with that of the regular part of the orbifold X.

Thank you for your attention.